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Anomalous Subvarieties—Structure Theorems and Applications

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When a fixed algebraic variety in a multiplicative group variety is intersected with the union of all algebraic subgroups of fixed dimension, a key role is played by what we call the anomalous subvarieties. These arise when the algebraic variety meets translates of subgroups in sets larger than expected. We prove a Structure Theorem for the anomalous subvarieties, and we give some applications, emphasizing in particular the case of codimension two. We also state some related conjectures about the boundedness of absolute height on such intersections as well as their finiteness.

1 Introduction

For $n \geq 1$ let $\mathcal{X}$ be an algebraic subvariety of the group variety $\mathbf{G}_m^n$ defined by the non-vanishing of the coordinates $x_1, \ldots, x_n$ in affine n-space. In this paper we are interested in the intersection of $\mathcal{X}$ with varying algebraic subgroups of $\mathbf{G}_m^n$ restricted only by dimension. Recall that every such subgroup is defined by monomial equations $x_1^{a_1} \ldots x_n^{a_n} = 1$, and its dimension is $n - h$, where h is the rank of the subgroup of $\mathbf{Z}^n$

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generated by the exponent vectors $(a_1, \dots, a_n)$. Thus the intersection describes the set of points at which given algebraic functions in several variables take values that satisfy a certain number of independent multiplicative relations.

In [4] we studied such intersections when $\mathcal{X} = \mathcal{C}$ is a curve defined over the field $\overline{\mathbf{Q}}$ of algebraic numbers, and we obtained two results: the first about the boundedness of the absolute height on the set, and the second about the finiteness of the set. The present paper concerns primarily the first of these results and its generalization to arbitrary $\mathcal{X}$. For the moment we do not prove any new results about bounded height, but we introduce a new set $\mathcal{X}^{oa}$ analogous to the set $\mathcal{X}^{o}$ defined by the first and third author in [8]. One of the main results of [8] is that $\mathcal{X}^{o}$ is Zariski-open in $\mathcal{X}$. The same authors then observed in [24] that this set occurs naturally in the study of heights. Namely, let $\mathcal{H}_1$ be the union of all algebraic subgroups of $\mathbf{G}_m^n$ with dimension 1. Then if $\mathcal{X}$ is defined over $\overline{\mathbf{Q}}$, the intersection $\mathcal{X}^{o} \cap \mathcal{H}_1$ is a set of bounded height. We will formulate a similar assertion as a conjecture involving $\mathcal{X}^{oa}$ and unions of higher-dimensional algebraic subgroups; and we will prove that $\mathcal{X}^{oa}$ too is Zariski-open in $\mathcal{X}$.

In fact it can happen that $\mathcal{X}^{oa}$ is empty, and then our conjecture says nothing. But we are able to refine our openness result to a geometrical "Structure Theorem" (Theorem 1.4), also analogous to an assertion in [8], which shows that $\mathcal{X}$ empties itself in an orderly way.

Our Structure Theorem can be used to give a characterization (Theorem 1.4) of the full set $\mathcal{X} \cap \mathcal{H}_1$. This generalizes an auxiliary result of [24], and it is the analogue of a well-known characterization (in the Manin-Mumford context) of the torsion points on $\mathcal{X}$; that is, the set $\mathcal{X} \cap \mathcal{H}_0$ where $\mathcal{H}_0$ is the union of all algebraic subgroups of $\mathbf{G}_m^n$ with dimension 0.

A related application of our Structure Theorem concerns a conjecture of Schinzel about lacunary polynomials. This conjecture was proved in [24] using the bounded height property for $\mathcal{X}^{o} \cap \mathcal{H}_1$ mentioned above. We are now able to give a quicker and more natural proof of this conjecture: and indeed in a stronger form (Theorem 1.5) involving unspecified roots of unity.

Finally we deduce from our Theorem 1.4 a finiteness result (Theorem 1.7) in the style of [4], involving a set $\mathcal{X}^{ta}$ analogous to the set $\mathcal{X}^{*}$, also introduced by the first and third author in [8].

We start by defining $\mathcal{X}^{oa}$. The work of [5], on curves defined over arbitrary fields of zero characteristic, shows that the restriction to $\overline{\mathbf{Q}}$ is inappropriate. So for the time being we will suppose only that our variety $\mathcal{X}$ is defined over the field $\mathbf{C}$ of complex numbers, and is over this field irreducible. For brevity we will refer to translates of algebraic subgroups as cosets.

Definition 1.1. An irreducible subvariety $\mathcal{Y}$ of $\mathcal{X}$ is anomalous (or better, $\mathcal{X}$-anomalous) if it has positive dimension and lies in a coset K in $\mathbf{G}_m^n$ satisfying

$$\dim K \ \leq \ n - \dim \mathcal{X} + \dim \mathcal{Y} - 1. \tag{1.1}$$

We remark at once that the definition remains unchanged when we require an equality in (1.1): if the inequality is strict, then it will become an equality after replacing K with a larger coset. This observation will be used repeatedly in the sequel.

The dimension conditions can be stated more succinctly as

$$\dim \mathcal{Y} \ > \ \max\{0, \dim \mathcal{X} + \dim K - n\}. \tag{1.2}$$

Notice that the quantity $\dim \mathcal{X} + \dim K - n$ on the right-hand side of (1.2) is what one would expect the dimension of $\mathcal{X} \cap K$ to be if $\mathcal{X}$ and K were in general position with non-empty intersection; and of course $\mathcal{Y}$ lies in this intersection. We will expand on this remark later.

Definition 1.2. The deprived set $\mathcal{X}^{oa}$ is what remains of $\mathcal{X}$ after removing all anomalous subvarieties.

In some sense this is not so far from $\mathcal{X}^o$ itself, which was defined in [8] (p.335) as what remains after removing all positive-dimensional cosets K contained in $\mathcal{X}$. In fact if $X \neq \mathbf{G}_m^n$, then (1.1) is implied by

$$\dim K \ \leq \ n - \dim \mathcal{X} + \dim \mathcal{Y} - (n - \dim \mathcal{X}) \ = \ \dim \mathcal{Y},$$

then in which case $\mathcal{Y}$ must be a component of K, and so we recover $\mathcal{X}^o$. Thus

$$\mathcal{X}^{oa} \ \subseteq \ \mathcal{X}^o \ \subseteq \ \mathcal{X}$$

(except when $\mathcal{X} = \mathbf{G}_m^n$).

Definition 1.3. An anomalous subvariety of $\mathcal{X}$ is maximal if it is not contained in a strictly larger anomalous subvariety of $\mathcal{X}$.

Clearly it is enough to remove these maximal anomalous subvarieties to obtain $\mathcal{X}^{oa}$.

To state the main result of this paper it is clearer to restrict to connected algebraic subgroups H, which we shall refer to in the usual way as tori.

The following is our main Structure Theorem.

Theorem 1.4. Let $\mathcal{X}$ be an irreducible variety in $\mathbf{G}_m^n$ of positive dimension defined over $\mathbf{C}$.

(a) For any torus H with

$$1 \leq h = n - \dim H \leq \dim \mathcal{X} \tag{1.3}$$

the union $\mathcal{Z}_H$ of all subvarieties $\mathcal{Y}$ of $\mathcal{X}$ contained in any coset K of H with

$$\dim \mathcal{Y} = \dim \mathcal{X} - h + 1 \tag{1.4}$$

is a closed subset of $\mathcal{X}$, and the product $H\mathcal{Z}_H$ is not dense in $\mathbf{G}_m^n$.

(b) There is a finite collection $\Phi = \Phi_{\mathcal{X}}$ of such tori H such that every maximal anomalous subvariety $\mathcal{Y}$ of $\mathcal{X}$ is a component of $\mathcal{X} \cap gH$ for some H in Φ satisfying (1.3) and (1.4) and some g in $\mathcal{Z}_H$; and $\mathcal{X}^{oa}$ is obtained from $\mathcal{X}$ by removing the $\mathcal{Z}_H$ for all H in Φ. In particular $\mathcal{X}^{oa}$ is open in $\mathcal{X}$. □

The openness of $\mathcal{X}^{oa}$ is the analogue of the first part of Theorem 1(a) of [8] (p.335) about $\mathcal{X}^o$. See also Theorems 1 and 2(i) of [23] (p.159) or Theorem 4.2.3(a) of [3] (p.95).

The assertion about maximal anomalous subvarieties is the analogue of the first part of the assertion which begins the second paragraph of section 5 of [8] (p.343) about maximal cosets in $\mathcal{X}^o$. See also [23] (p.166) or Theorem 3.3.9(b) of [3] (p.91). As there, the maximality is vital for the validity, because a maximal anomalous subvariety can be intersected with a generic equation $x_1^{a_1} \dots x_n^{a_n} = \lambda$ to give a non-maximal anomalous subvariety which cannot be controlled in terms of a finite collection of subgroups H.

The claim about $H\mathcal{Z}_H$ can be regarded as a condition on the set of elements g with $K = gH$ occurring in (1.4). This may become clearer if we normalize H as follows. Any torus H of dimension $n-h$ is isomorphic to $\mathbf{G}_m^{n-h}$; more precisely, there is an automorphism α_H of $\mathbf{G}_m^n$ such that $\alpha_H(H) = \{1\}^h \times \mathbf{G}_m^{n-h}$ (which we identify with $\mathbf{G}_m^{n-h}$). See for example [23] or Chapter 3 of [3]. Denote by π_h the projection of $\mathbf{G}_m^n$ to $\mathbf{G}_m^h$ (which we identify with $\mathbf{G}_m^h \times \{1\}^{n-h}$). Now $\mathcal{Z}_H$ is made up of various $\mathcal{Y}$ in their gH, so $U_H = \pi_h(\alpha_H(\mathcal{Z}_H))$ is made up of various $\pi_h(\alpha_H(\mathcal{Y}))$ in their $\pi_h(\alpha_H(gH))$; these latter however are the single points $\pi_h(\alpha_H(g))$. So the g are characterized by the fact that the $\pi_h(\alpha_H(g)) = \pi_h(\alpha_H(\mathcal{Y}))$ constitute the set U_H. As $\alpha_H(\mathcal{Z}_H))$ lies in $U_H \times \mathbf{G}_m^{n-h}$, we see that $\alpha_H(H\mathcal{Z}_H)) = U_H \times \mathbf{G}_m^{n-h}$; thus the claim about $H\mathcal{Z}_H$ means just that U_H is not dense in $\mathbf{G}_m^h$.

Clearly $\mathcal{Z}_H$ lies in $\mathcal{X}\cap\alpha_H^{-1}(U_H\times\mathbf{G}_m^{n-h})$ and so $\mathcal{X}\setminus\mathcal{X}^{oa}$ lies in the union of $\mathcal{X}\cap\alpha_H^{-1}(U_H\times\mathbf{G}_m^{n-h})$ taken over all H in the finite set Φ. This resembles the displayed equation near the end of the third paragraph of section 5 of [8] (p.343), which in our notation would say that $\mathcal{X}\setminus\mathcal{X}^o$ is a finite union of $\alpha_H^{-1}(\mathcal{V}_H\times\mathbf{G}_m^{n-h})$ for closed $\mathcal{V}_H$ in $\mathbf{G}_m^h$. See also Theorem B(a) of [6] (p.2250) or Remark 4.2.4 of [3] (p.95). But such a precise description of $\mathcal{X}\setminus\mathcal{X}^{oa}$ is not so simple: firstly our U_H are only constructible rather than closed, secondly we need the extra intersection with $\mathcal{X}$, and thirdly $\mathcal{Z}_H$ can be strictly smaller than $\mathcal{X}\cap\alpha_H^{-1}(U_H\times\mathbf{G}_m^{n-h})$.

Since the main work for this paper was done, we have become aware of some work [25] of Zilber. We will discuss the conjectural aspects of this later (section 5), but it also contains techniques related to our proof of Theorem 1.4. Indeed Corollary 3 of [25] (p.37) is a version of our Proposition 3.1 (section 3). We have decided to present our own proof because it is effective in nature and, furthermore, in principle it can be used to derive explicit estimates for $\mathcal{X}^{oa}$ of the same nature as those already existing for $\mathcal{X}^o$. For example, if $\mathcal{X}$ is defined by equations of degree at most $D\geq 1$, then the H in our Φ can be defined by equations $x_1^{a_1}\dots x_n^{a_n}=1$ with $\max\{|a_1|,\dots,|a_n|\}\leq cD^\kappa$, where c and κ depend only on the ambient dimension n. A similar bound for $\mathcal{X}^o$ (with $\kappa=1$) is implicit in the proof of Theorem 1.4 of [23] (p.159). See also [3] (p.89). In fact this type of quantitative consideration plays a key role even in the proof of our non-quantitative Theorem 1, and we will discuss it later more efficiently in terms of geometric degrees. It has a certain uniformity aspect, being independent of the coefficients in the equations defining $\mathcal{X}$, and it leads to what may be called a Uniform Structure Theorem (see section 3).

Using standard effective elimination theory it is also possible to show that the closed sets $\mathcal{Z}_H$ can be defined by equations of degree at most cD^κ, but we omit the details. An analogue for $\mathcal{X}^o$ (also with $\kappa=1$) is Theorem 2(i) of [23] (p.159). See also [3] (p.95).

Next we state our conjecture about bounded height. We need a height function on the set $\mathbf{G}_m^n(\overline{\mathbf{Q}})$; the precise choice is unimportant but for definiteness we use

$$h(x)\;=\;h(\xi_1)+\dots+h(\xi_n)$$

for $x=(\xi_1,\dots,\xi_n)$, where $h(\xi)$ denotes the absolute (logarithmic) Weil height. Denote by $\mathcal{H}_d$ the union of all algebraic subgroups of $\mathbf{G}_m^n$ with dimension d.

Bounded Height Conjecture. *Let $\mathcal{X}$ be an irreducible variety in $\mathbf{G}_m^n$ of dimension r defined over $\overline{\mathbf{Q}}$. Then $\mathcal{X}^{oa}\cap\mathcal{H}_{n-r}$ is a set of bounded height.*

When $\mathcal{X}$ is a curve $\mathcal{C}$, then (1.1) shows that $\mathcal{X}^{oa}=\mathcal{C}$ when $\mathcal{C}$ is not contained in a coset of dimension $n-1$ (and empty otherwise). So here the Bounded Height Conjecture reduces to Theorem 1 of [4] (p.1120). When $\mathcal{X}$ is a hypersurface, then (1.1) shows that

$\mathcal{X}^{oa} = \mathcal{X}^{o}$. So here the Bounded Height Conjecture reduces to Theorem 1 of [24] (p.524). Nothing else is known. In a forthcoming paper [7] we shall prove the Bounded Height Conjecture for any plane in any $\mathbf{G}_m^n$.

The second main result of this paper is the following.

Theorem 1.5. Let $\mathcal{X}$ be an irreducible variety in $\mathbf{G}_m^n$ defined over $\overline{\mathbf{Q}}$. Then there exists a finite collection $\Psi = \Psi_{\mathcal{X}}$ of translates T of tori by torsion points, satisfying

$$\dim(\mathcal{X} \cap T) \geq \dim T - 1,$$

such that $\mathcal{X} \cap \mathcal{H}_1$ is the union of the $(\mathcal{X} \cap T) \cap \mathcal{H}_1$ for all T in Ψ. □

This refines the auxiliary Theorem 2 of [24] (p.530), which was also restricted to the intersection of $\mathcal{X}$ with the union of all one-dimensional tori. The present version treats the larger union of all one-dimensional algebraic groups. If instead $\mathcal{X}$ is intersected with the union of all zero-dimensional algebraic groups, then we obtain a classical situation analogous to that covered by conjectures of Manin and Mumford for subvarieties of abelian varieties (see for example [14] pp.220, 221). In the multiplicative case the analogous conjectures (and much more) were proved by Laurent [16]. His Theorem 2 (p.307) implies the existence of a finite collection of translates T of tori by torsion points, satisfying $\dim(\mathcal{X} \cap T) \geq \dim T$, such that $\mathcal{X} \cap \mathcal{H}_0$ is the union of the $(\mathcal{X} \cap T) \cap \mathcal{H}_0$. Of course the dimension inequality here means just that T is contained in $\mathcal{X}$, so that $(\mathcal{X} \cap T) \cap \mathcal{H}_0 = T \cap \mathcal{H}_0$.

Thus for example the set of torsion points of $\mathcal{X}$ is nearly a group in the sense that it is a finite union of translated groups. The set $\mathcal{X} \cap \mathcal{H}_1$ probably has no such structure. But if $\mathcal{X}_{n-1}$ is a hypersurface in $\mathbf{G}_m^n$ then $\mathcal{X}_{n-1} \cap \mathcal{H}_1$ is infinite, and, as we have seen, the heights of its points are usually bounded above. Our present Theorem 1.4 at least reduces the problem of describing $\mathcal{X} \cap \mathcal{H}_1$ for general $\mathcal{X}$ to this hypersurface case in that we can regard $\mathcal{X} \cap T$ as a hypersurface in T, and this T is more or less the same as some $\mathbf{G}_m^d$. The analogous description of $\mathcal{X} \cap \mathcal{H}_2$ seems out of reach at present.

Next we give the consequence for lacunary polynomials with algebraic coefficients.

Theorem 1.6. For $n \geq 2$ let P and Q be coprime polynomials in n variables defined over $\overline{\mathbf{Q}}$. Then there exists $B = B(P, Q)$ depending only on P and Q with the following property. Suppose $\zeta_1, \ldots, \zeta_n$ are roots of unity, $a_1, \ldots, a_n$ are rational integers, and τ is a non-zero complex number with

$$P(\zeta_1\tau^{a_1},\dots,\zeta_n\tau^{a_n}) = Q(\zeta_1\tau^{a_1},\dots,\zeta_n\tau^{a_n}) = 0.$$

Then there exist rational integers $b_1,\dots,b_n$ with

$$0 < \max\{|b_1|,\dots,|b_n|\} \leq B,\ (\zeta_1\tau^{a_1})^{b_1}\dots(\zeta_n\tau^{a_n})^{b_n} = 1.$$

In particular, if τ is not a root of unity, then $\zeta_1^{b_1}\dots\zeta_n^{b_n} = 1$ and $a_1b_1+\cdots+a_nb_n = 0$. □

In the special case $\zeta_1 = \cdots = \zeta_n = 1$, the existence of a non-trivial relation $a_1b_1+\cdots+a_nb_n = 0$ between the exponents $a_1,\dots,a_n$ had been proposed by Schinzel as Conjecture 1 of [22] (p.298) after he proved it for $n = 3$ (with P and Q over any field) as Theorem 1 of [21] (p.47). These results are foreshadowed in Lemma 8 of [20] (p.135). The general conjecture was proved by the first and third authors in [24]. For unrestricted $\zeta_1,\dots,\zeta_n$ even the case $n = 3$ here is new. The reader may consult [22], especially sections 6.2 and 6.3, to see the implications of such results for irreducibility.

In principle there is no difficulty in calculating a value for the above constant $B(P,Q)$. But in contrast to the geometrical estimates available for Theorem 1.4, this constant cannot depend only on n and the degrees of P and Q. As already observed by Schinzel, the example

$$P(x_1,x_2) = x_1 - 2,\ Q(x_1,x_2) = x_2 - 2^a$$

with $n = 2$ and $(\zeta_1\tau^{a_1},\zeta_2\tau^{a_2}) = (2,2^a)$ has $B(P,Q) \geq a$, so it must depend also on the coefficients of P and Q.

Finally we state our finiteness result, postponing the definition of $\mathcal{X}^{ta}$ to section 5.

Theorem 1.7. For $n \geq 2$ let $\mathcal{X}$ be an irreducible variety in $\mathbf{G}_m^n$ of dimension $n-2$ defined over $\overline{\mathbf{Q}}$. Then $\mathcal{X}^{ta}$ is Zariski-open in $\mathcal{X}$ and $\mathcal{X}^{ta}\cap\mathcal{H}_1$ is a finite set. □

See section 5 also for further comments about this result.

The plan of our paper is as follows.

In section 2 we prepare the way for the proof of Theorem 1.4. A crucial ingredient is the classical result of Ax amounting to the analogue of Schanuel's Conjecture in fields of complex power series in several variables. In fact it suffices to use an earlier result due to Chabauty. Throughout this section we decided for the sake of clarity to present several of the arguments in all details.

Then we give in section 3 the proof of Theorem 1.4, using what may be regarded as an effective refinement of Chabauty's Theorem, and maintaining the somewhat detailed style of exposition, especially with regard to uniformity aspects.

After that in section 4 we prove Theorems 1.4 and 1.5. In fact we do Theorem 1.5 first, making essential use of certain height lower bounds, and then we deduce Theorem 1.4 from it. The arguments of [24] ran in the opposite direction. We have chosen this variant in order to highlight the essential equivalence of the two points of view.

In section 5 we discuss further the Bounded Height Conjecture, and we supplement it with more conjectures. One of these, the Torsion Finiteness Conjecture, generalizes to arbitrary varieties and also sharpens the finiteness result of [4]. It is already known for hypersurfaces defined over $\overline{\mathbf{Q}}$. We will show that our Theorem 1.4 implies our Theorem 1.7 and in particular the Torsion Finiteness Conjecture for varieties in $\mathbf{G}_m^n$ of dimension $n-2$ defined over $\overline{\mathbf{Q}}$. We also indicate the links with the work of Zilber and some more recent work of Pink [18].

Finally in a short section 6 we make some comments about contexts more general than the multiplicative group variety $\mathbf{G}_m^n$. Our Theorem 1.4 should probably admit an extension to subvarieties $\mathcal{X}$ of any semiabelian variety (for which our methods of proof should carry over with relatively routine modifications). A semiabelian extension of the Bounded Height Conjecture still seems plausible. However in the context of mixed Shimura varieties, emphasized by Pink, the natural analogue of the Bounded Height Conjecture is false. As for possible semiabelian versions of our Theorems 1.4, 1.5 and 1.7, these probably cannot be proved with our methods owing to the use of height lower bounds, whose extensions at present need some extra complex multiplication hypotheses. For these reasons we decided to restrict the present paper to subvarieties of $\mathbf{G}_m^n$; further the accompanying estimates then do not involve extra considerations such as polarizations.

2 Preliminaries on Jacobian matrices

We will make constant and crucial use of the following classical result on the dimension of the fibers of a morphism, in which $\dim_v \mathcal{V}$ denotes the maximum dimension of the components of $\mathcal{V}$ through v, and the dimension of the empty set is to be interpreted as -1.

Fiber Dimension Theorem (FDT). *Let φ be a dominant morphism from the irreducible variety $\mathcal{V}$ to the irreducible variety $\mathcal{W}$. Then*

(a) *For all* v *in* $\mathcal{V}$, *we have* $\dim_v \varphi^{-1}(\varphi(v)) \geq \dim \mathcal{V} - \dim \mathcal{W}$; *in particular, for all* w *in* $\mathcal{W}$ *every (non-empty) component of the fiber* $\varphi^{-1}(w)$ *has dimension at least* $\dim \mathcal{V} - \dim \mathcal{W}$.

(b) *There exists an open dense subset* U *in* $\mathcal{W}$ *such that for every* w *in* U *we have* $\dim \varphi^{-1}(w) = \dim \mathcal{V} - \dim \mathcal{W}$.

(c) *For every integer* k, *the set* $\mathcal{V}_k$ *of all* v *in* $\mathcal{V}$ *such that* $\dim_v \varphi^{-1}(\varphi(v)) \geq k$ *is closed in* $\mathcal{V}$.

Proof. See, for example, Danilov's article [11]. The first part of our (a) is just (a) of the first Theorem (p.228), and the second part of our (a) follows immediately by taking any v on the component in question not lying on any other component. Our (b) is just (b) of the same Theorem. Finally our (c) is just a special case of the second Theorem (p.228), described there as Chevalley's semicontinuity theorem. ■

If desired, it is not difficult to make (c) effective. When $\mathcal{V}, \mathcal{W}$ are in $\mathbf{G}_m^n$, and these together with φ are defined by equations of degrees at most $D \geq 1$, then standard elimination theory shows that the $\mathcal{V}_k$ can be defined by equations of degree at most cD^κ, where c and κ depend only on n. We omit the details.

Any variety $\mathcal{X}$ in affine n-space, irreducible over $\mathbf{C}$, has a canonical Chow ideal $\mathcal{I}(\mathcal{X})$ in $\mathbf{C}[x_1, \ldots, x_n]$. This ideal even comes with a canonical basis $P_1, \ldots, P_N$, obtained by substituting linear polynomials with coefficients taken from generic skew-symmetric matrices into the Chow form; and its corresponding variety is precisely $\mathcal{X}$ (see for example [12] p.51 for the projective case). In fact the prime ideal $\mathcal{P}(\mathcal{X})$ of $\mathcal{X}$ is the unique isolated primary component of $\mathcal{I}(\mathcal{X})$; see for example Lemma 11 of [17] (p.251). We denote by $J(\mathcal{X})$ the Jacobian matrix with N rows and n columns with entry $\frac{\partial P_i}{\partial x_j}$ in the ith row and jth column ($i = 1, \ldots, N; j = 1, \ldots, n$).

We will add rows to $J(\mathcal{X})$ as follows. For $\mathbf{z} = (z_1, \ldots, z_n)$ in $\mathbf{C}^n$ we form the row

$$r(\mathbf{z}) = \left(\frac{z_1}{x_1}, \ldots, \frac{z_n}{x_n} \right).$$

Then for $h \geq 1$ and $\mathbf{z}_1, \ldots, \mathbf{z}_h$ in $\mathbf{C}^n$ we define $J(\mathbf{z}_1, \ldots, \mathbf{z}_h; \mathcal{X})$ as the matrix with $N + h$ rows and n columns obtained by adjoining the rows $r(\mathbf{z}_1), \ldots, r(\mathbf{z}_h)$ to $J(\mathcal{X})$.

The entries now lie in the field $\mathbf{C}(x_1, \ldots, x_n)$, and so we can consider the rank of these matrices. But also if $\mathcal{Y}$ is any irreducible variety in $\mathbf{G}_m^n$, we can think of the entries in the function field $\mathbf{C}(\mathcal{Y})$, and then we denote the rank by $\mathrm{rank}_{\mathcal{Y}}$.

In case $\mathbf{a}$ lies in $\mathbf{Z}^n$, we define $x^{\mathbf{a}} = x_1^{a_1} \dots x_n^{a_n}$. And for $\mathbf{a}_1, \dots, \mathbf{a}_h$ in $\mathbf{Z}^n$ we define a map $\varphi = \varphi(\mathbf{a}_1, \dots, \mathbf{a}_h)$, depending on $\mathbf{a}_1, \dots, \mathbf{a}_h$, which takes $(x_1, \dots, x_n)$ in $\mathbf{G}_m^n$ to $(x^{\mathbf{a}_1}, \dots, x^{\mathbf{a}_h})$ in $\mathbf{G}_m^h$.

We denote the dimension of $\mathcal{X}$ by r.

Lemma 2.1. Suppose $\mathbf{a}_1, \dots, \mathbf{a}_h$ in $\mathbf{Z}^n$ are such that $\operatorname{rank}_{\mathcal{X}} J(\mathbf{a}_1, \dots, \mathbf{a}_h; \mathcal{X}) \leq n - r + h - 1$. Then $x^{\mathbf{a}_1}, \dots, x^{\mathbf{a}_h}$ are algebraically dependent on $\mathcal{X}$. □

Proof. The rank condition means that there are at least $(N+h)-(n-r+h-1) = N-n+r+1$ relations

$$Q_1 r(\mathbf{a}_1) + \dots + Q_h r(\mathbf{a}_h) + R_1 r_1 + \dots + R_N r_N \ = \ 0, \tag{2.1}$$

$\mathbf{C}(\mathcal{X})$-linearly independent, with coefficients $Q_1, \dots, Q_h, R_1, \dots, R_N$ in $\mathbf{C}(\mathcal{X})$; here $r_1, \dots, r_N$ are the rows of $J(\mathcal{X})$.

Suppose, on the contrary, that $\varphi_1 = x^{\mathbf{a}_1}, \dots, \varphi_h = x^{\mathbf{a}_h}$ are algebraically independent on $\mathcal{X}$. Then $\mathbf{C}(\varphi_1, \dots, \varphi_h)$ is a purely transcendental subfield of $\mathbf{C}(\mathcal{X})$. We may therefore extend the derivations $\frac{\partial}{\partial \varphi_1}, \dots, \frac{\partial}{\partial \varphi_h}$ on $\mathbf{C}(\varphi_1, \dots, \varphi_h)$ to $\mathbf{C}(\mathcal{X})$. Applying $\frac{\partial}{\partial \varphi_l}$ ($l = 1, \dots, h$) to $P_i(x_1, \dots, x_n) = 0$ ($i = 1, \dots, N$), we deduce that $r_i v_l = 0$, where v_l is the column with entries $\frac{\partial x_j}{\partial \varphi_l}$ ($j = 1, \dots, n$). Multiplying (2.1) by v_l gives

$$Q_1 r(\mathbf{a}_1) v_l + \dots + Q_h r(\mathbf{a}_h) v_l \ = \ 0 \ (l = 1, \dots, h). \tag{2.2}$$

Now for any $\mathbf{a} = (a_1, \dots, a_n)$ in $\mathbf{Z}^n$ we have

$$\frac{\partial (x^{\mathbf{a}})}{\partial \varphi_l} \ = \ x^{\mathbf{a}} \sum_{j=1}^{n} \frac{a_j}{x_j} \frac{\partial x_j}{\partial \varphi_l} \ = \ x^{\mathbf{a}} r(\mathbf{a}) v_l \ (l = 1, \dots, h). \tag{2.3}$$

Taking $\mathbf{a} = \mathbf{a}_m$ we see that this is 1 if $m = l$ and 0 if $m \neq l$. Thus in (2.2) the only survivor is $Q_l x^{-\mathbf{a}_l}$; consequently (2.1) reduces to $R_1 r_1 + \dots + R_N r_N = 0$. These $N - n + r + 1$ relations remain independent, showing that $\operatorname{rank}_{\mathcal{X}} J(\mathcal{X}) \leq N - (N - n + r + 1) = n - r - 1$.

Now if the Chow ideal $\mathcal{I}(X)$ were exactly the prime ideal $\mathcal{P}(X)$ of $\mathcal{X}$, then this inequality would contradict the standard Jacobian criterion, because the codimension of $\mathcal{X}$ is $n - r$. In general, because $\mathcal{P}(X)$ is the unique isolated primary component of $\mathcal{I}(X)$, there is P not vanishing on $\mathcal{X}$ such that $\mathcal{P}(X)$ lies in $P^{-1}\mathcal{I}(X)$, and so the same inequality would follow for the rank of the matrix formed with generators of $\mathcal{P}(X)$, giving the same contradiction. ■

In this situation we see that the image $\varphi(\mathbf{a}_1, \ldots, \mathbf{a}_h)(\mathcal{X})$ in $\mathbf{G}_m^h$ has dimension at most $h-1$.

For $\mathbf{a}_1, \ldots, \mathbf{a}_h$ in $\mathbf{Z}^n$ we define $H(\mathbf{a}_1, \ldots, \mathbf{a}_h)$ by the equations $x^{\mathbf{a}_1} = \cdots = x^{\mathbf{a}_h} = 1$. This is an algebraic subgroup of $\mathbf{G}_m^n$. If $\mathbf{a}_1, \ldots, \mathbf{a}_h$ are $\mathbf{Q}$-linearly independent, then its dimension is $n-h$. See for example Proposition 3.2.7 of [3] (p.83).

Lemma 2.2. Suppose $\mathcal{Y}$ of dimension s in $\mathcal{X}$ is $\mathcal{X}$-anomalous and lies in a translate of $H(\mathbf{a}_1, \ldots, \mathbf{a}_h)$ with $h = r-s+1$. Then $\mathrm{rank}_{\mathcal{Y}} J(\mathbf{a}_1, \ldots, \mathbf{a}_h; \mathcal{X}) \leq n-r+h-1$. □

Proof. There are $\alpha_1, \ldots, \alpha_h$ in $\mathbf{C}$ such that the functions $x^{\mathbf{a}_1} - \alpha_1, \ldots, x^{\mathbf{a}_h} - \alpha_h$ vanish on the translate in question. So these together with $P_1, \ldots, P_N$ lie in the prime ideal $\mathcal{P}(Y)$ of $\mathcal{Y}$. Consequently the Jacobian of all the functions has rank at most $n-s = n-r+h-1$. The lemma follows using the identities

$$\frac{\partial(x^{\mathbf{a}} - \alpha)}{\partial x_i} = x^{\mathbf{a}} \frac{a_i}{x_i} \; (i = 1, \ldots, n)$$

analogous to (2.3). ■

Lemma 2.3. Suppose $\mathcal{Y}$ of dimension s in $\mathcal{X}$ is maximal $\mathcal{X}$-anomalous and lies in a translate of $H(\mathbf{a}_1, \ldots, \mathbf{a}_h)$ with $h = r-s+1$. Then the image $\varphi(\mathbf{a}_1, \ldots, \mathbf{a}_h)(\mathcal{X})$ has dimension at least $h-1$. □

Proof. Under $\varphi(\mathbf{a}_1, \ldots, \mathbf{a}_h) = \varphi$ the translate goes into a single point w. If the dimension of $\varphi(\mathcal{X})$ were strictly less than $h-1$, then by FDT(a), every component of $\mathcal{X} \cap \varphi^{-1}(w)$ would have dimension strictly greater than $r-(h-1) = s$. Each such component remains $\mathcal{X}$-anomalous. But $\mathcal{Y}$ lies in one of these components, contradicting the maximality.

Lemma 2.4. Suppose there exists $\mathbf{z} \neq 0$ in $\mathbf{C}^n$ such that $\mathrm{rank}_{\mathcal{X}} J(\mathbf{z}; \mathcal{X}) \leq n-r$. Then $\mathcal{X}$ is $\mathcal{X}$-anomalous. □

Proof. If the expression $x^{\mathbf{z}}$ made sense, and if Lemma 2.1 (with $h=1$) could be applied, then we could conclude that $x^{\mathbf{z}}$ is constant on $\mathcal{X}$. The work [1] of Ax would then give non-zero $\mathbf{a}$ in $\mathbf{Z}^n$ with $x^{\mathbf{a}}$ constant on $\mathcal{X}$, showing that $\mathcal{X}$ is $\mathcal{X}$-anomalous.

A proper proof is not difficult. Because $\mathrm{rank}_{\mathcal{X}} J(\mathcal{X})$ is already $n-r$, we deduce a relation $r(\mathbf{z}) = R_1 r_1 + \cdots + R_N r_N$ in the notation of (2.1). Denoting by $\delta_1, \ldots, \delta_r$ independent derivations on $\mathbf{C}(\mathcal{X})$, we deduce from $P_i = 0$ $(i = 1, \ldots, N)$ that

$$r_i v_l = 0 \; (l = 1, \ldots, r), \tag{2.4}$$

where now v_l is the column with entries $\delta_l(x_j)$ $(j = 1, \ldots, n)$. Thus also

$$r(\mathbf{z})v_l \ = \ 0 \ (l = 1, \ldots, r). \tag{2.5}$$

There are now several standard ways of proceeding. For example one can select generic linear polynomials $y_1, \ldots, y_r$ in $x_1, \ldots, x_n$, and use $\delta_l = \frac{\partial}{\partial y_l}$ $(l = 1, \ldots, r)$. Around a point x on $\mathcal{X}$ with $y_1 = \cdots = y_r = 0$ we have formal expansions $x_j = \xi_j \exp(X_j)$ with X_j power series in $y_1, \ldots, y_r$ with no constant terms $(j = 1, \ldots, n)$. Now (2.5) implies that the power series $z_1X_1 + \cdots + z_nX_n$ is locally constant near x. Thus the transcendence degree t over $\mathbf{C}$ of $\mathbf{C}(X_1, \ldots, X_n, \exp(X_1), \ldots, \exp(X_n))$ is at most $(n-1) + \dim \mathcal{X} = n - 1 + r$.

If $X_1, \ldots, X_n$ are linearly dependent over $\mathbf{Q}$, then we get at once a non-trivial monomial constant on $\mathcal{X}$, so the lemma is proved. Otherwise, Corollary 1 of [1] (p.253), which is a power series analogue of Schanuel's Conjecture over $\overline{\mathbf{Q}}$, shows that $t \geq n + r'$, where r' is the rank of the matrix with entries $\frac{\partial X_j}{\partial y_l} = \frac{1}{x_j}\frac{\partial x_j}{\partial y_l}$ $(j = 1, \ldots, n; l = 1, \ldots, r)$. Because $x_1, \ldots, x_n$ locally parameterize $\mathcal{X}$, we see that $r' = r$, and so this leads to the desired contradiction.

In alternative language one can deduce from (2.5) a relation $z_1 \mathrm{d}x_1/x_1 + \cdots + z_n \mathrm{d}x_n/x_n = 0$ on the differentials of $\mathcal{X}$, which by integration yields a relation $z_1 \log(x_1/\xi_1) + \cdots + z_n \log(x_n/\xi_n) = 0$ holding in any neighborhood of any non-singular point $(\xi_1, \ldots, \xi_n)$ of $\mathcal{X}$. This defines in the terminology of [1] (p.263) a "μ-variety" M of dimension $n-1$ containing $W = \mathcal{X}$. So we can apply the result of Chabauty in the form quoted by Ax to $I = \mathcal{X}$. We obtain an "algebraic μ-variety" A, or just coset, containing $\mathcal{X}$, with

$$\dim A \ \leq \ \dim M + \dim W - \dim I \ = \ n - 1. \tag{2.6}$$

This A is defined by some non-trivial $x^{\mathbf{a}}$ being constant, and as above we get what we want. ∎

It is this version due to Chabauty that we will make effective. In fact there are comparatively elementary proofs, for example, by considering the sequence of products $\mathcal{X}, \mathcal{X}\mathcal{X}, \mathcal{X}\mathcal{X}\mathcal{X}, \ldots$ or by using valuations. We will use the latter method in Lemma 3.2 below to give a good estimate for $\mathbf{a}$ in geometric terms.

Lemma 2.5. Suppose $\mathcal{Y}$ of dimension s in $\mathcal{X}$ is $\mathcal{X}$-anomalous and lies in a translate K of $H(\mathbf{a}_1, \ldots, \mathbf{a}_h)$ with $h = r - s + 1$ but does not lie in the singular locus of $\mathcal{X}$. Suppose further that $\mathbf{a}_1, \ldots, \mathbf{a}_h$ are $\mathbf{Q}$-linearly independent, that $x^{\mathbf{a}_1}, \ldots, x^{\mathbf{a}_h}$ have transcendence degree $h - 1$ on $\mathcal{X}$, and that the point $\varphi(\mathbf{a}_1, \ldots, \mathbf{a}_h)(K)$ is non-singular on $\varphi(\mathbf{a}_1, \ldots, \mathbf{a}_h)(\mathcal{X})$. Then there exists $\mathbf{z} \neq 0$ in $\mathbf{C}^n$ such that $\mathrm{rank}_{\mathcal{Y}}\, J(\mathbf{z}; \mathcal{X}) \leq n - r$. □

Proof. For $\varphi(\mathbf{a}_1,\ldots,\mathbf{a}_h)=\varphi$ the variety $\varphi(\mathcal{X})$ has dimension $h-1$. So it is defined by a single polynomial equation $F=0$ in $\mathbf{G}_m^h$. By Proposition 3 of [13] (p.188), we can find independent derivations $\delta_1,\ldots,\delta_r$ on $\mathbf{C}(\mathcal{X})$ such that the $\delta_l(x_j)$ $(l=1,\ldots,r;\ j=1,\ldots,n)$ are regular off the singular locus $\mathcal{X}_{\text{sing}}$ of $\mathcal{X}$. As earlier, write for brevity $\varphi_1=x^{\mathbf{a}_1},\ldots,\varphi_h=x^{\mathbf{a}_h}$, and let v_l $(l=1,\ldots,r)$ be the columns with entries $\delta_l(x_j)$ $(j=1,\ldots,n)$. Using F_k $(k=1,\ldots,h)$ for the partial derivatives, and applying δ_l $(l=1,\ldots,r)$ to $F(\varphi_1,\ldots,\varphi_h)=0$ we obtain after using the analogue $\delta_l(\varphi_k)=\varphi_k r(\mathbf{a}_k)v_l$ of (2.3) the equations

$$\sum_{k=1}^{h}F_k(\varphi_1,\ldots,\varphi_h)\varphi_k r(\mathbf{a}_k)v_l \;=\; 0\ (l=1,\ldots,r)$$

holding on $\mathcal{X}$. Because $\mathcal{Y}$ does not lie in $\mathcal{X}_{\text{sing}}$, we may specialize to $\mathcal{Y}$ in K. Writing $w=\varphi(K)=(w_1,\ldots,w_h)$, we find exactly the equations (2.5) with

$$\mathbf{z} \;=\; \sum_{k=1}^{h}F_k(w)w_k\mathbf{a}_k$$

holding now on $\mathcal{Y}$. We note that $\mathbf{z}\neq 0$ because $\mathbf{a}_1,\ldots,\mathbf{a}_h$ are linearly independent over $\mathbf{Q}$, therefore also over $\mathbf{C}$, and $F_1(w),\ldots,F_h(w)$ are not all zero by the non-singularity hypothesis. The equations (2.4) on $\mathcal{X}$ continue to hold on $\mathcal{Y}$, and so we get the matrix equations $J(\mathbf{z};\mathcal{X})v_l=0$ $(l=1,\ldots,r)$ on $\mathcal{Y}$. Finally the independence of $\delta_1,\ldots,\delta_r$ implies the linear independence of $v_1,\ldots,v_r$, and this leads to the upper bound for the $\text{rank}_{\mathcal{Y}}$ required in the present lemma. ■

3 Proof of Theorem 1.4

We will use degree theory in $\mathbf{G}_m^n$ arising from the natural embedding in projective $\mathbf{P}_n$. We note that if $\mathcal{X}$ has dimension r and degree Δ, then the canonical basis elements of the Chow ideal have degrees at most $(r+1)\Delta$ (see [12] p.51). In fact, if we give up the Chow ideal, then we can get an improved upper bound Δ (see for example the arguments of [11] p.277 or [8] p.343). We will also use freely some simple versions of the Bézout Theorem (see for example the Theorem (p.251) of [11]).

The following result is the key estimate for $\mathcal{X}$-anomalous varieties $\mathcal{Y}$, which is uniform in $\mathcal{X}$ of fixed degree. For $\mathbf{a}=(a_1,\ldots,a_n)$ in $\mathbf{Z}^n$ we use the norm $|\mathbf{a}|=\max\{|a_1|,\ldots,|a_n|\}$.

Proposition 3.1. Given r with $1\leq r\leq n$ there are constants $c(n,r)$ and $\mu(n,r)$, depending only on n and r, with the following property. Suppose $\mathcal{X}$ in $\mathbf{G}_m^n$ is irreducible with dimension r and degree Δ. Then for every $\mathcal{X}$-anomalous variety $\mathcal{Y}$ there is $\mathbf{a}$ in $\mathbf{Z}^n$, with $x^{\mathbf{a}}$ constant on $\mathcal{Y}$ and

$$0 \ < \ |\mathbf{a}| \ \leq \ c(n,r)\Delta^{\mu(n,r)}. \qquad \square$$

Before starting on the proof, we settle the case $\mathcal{Y} = \mathcal{X}$.

Lemma 3.2. There is a constant $c_0(n)$, depending only on n, with the following property. Suppose $\mathcal{X}$ is $\mathcal{X}$-anomalous of positive dimension and degree Δ. Then there is $\mathbf{a}$ in $\mathbf{Z}^n$, with $x^{\mathbf{a}}$ constant on $\mathcal{X}$ and

$$0 \ < \ |\mathbf{a}| \ \leq \ c_0(n)\Delta^{n-1}. \qquad \square$$

Proof. We use the notion of a "field with a proper set of absolute values satisfying a product formula" in the sense of [5] (p.452). It is well known that a purely transcendental extension $\mathbf{C}(y_1, \ldots, y_r)$ is such a field (the valuations corresponding to polynomials irreducible over $\mathbf{C}$ taken modulo constants, together with one for the total degree). There is a logarithmic height function h on this field, which can be normalized to take non-negative values in $\mathbf{Z}$ with $h(y_1) = \cdots = h(y_r) = 1$. Here the "zero height group" Z in the sense of [5] (p.454) consists of non-zero elements of $\mathbf{C}$.

We find a copy L of this field in $\mathbf{C}(\mathcal{X})$ by taking $y_1, \ldots, y_r$ as generic linear polynomials in $x_1, \ldots, x_n$. Then $\mathbf{C}(\mathcal{X})$ is an algebraic extension of L with degree exactly Δ. Choose an embedding of $\mathbf{C}(\mathcal{X})$ in an algebraic closure $\overline{L}$ of L. There is a logarithmic height function h on $\overline{L}$ extending the h above, so that by restriction we get a height on $\mathbf{C}(\mathcal{X})$ itself.

We proceed to verify that

$$h(z) \ \geq \ \frac{1}{\Delta} \tag{3.1}$$

for all z in $\mathbf{C}(\mathcal{X})$ not in $\mathbf{C}$. This is an analogue of the Lehmer Problem for number fields, and it shows at the same time that the zero height group of $\mathbf{C}(\mathcal{X})$ remains Z. ∎

We start by noting that for any elementary symmetric function w of any elements $z_1, \ldots, z_m$ in $\overline{L}$ we have

$$h(w) \ \leq \ h(z_1) + \cdots + h(z_m). \tag{3.2}$$

This is easily checked using the obvious estimate

$$\max\{1, |w|\} \le \max\{1, |z_1|\} \cdots \max\{1, |z_m|\}$$

for any ultrametric valuation.

Now any z in $\mathbf{C}(\mathcal{X})$ has $m \le \Delta$ conjugates $z_1, \ldots, z_m$ in $\overline{L}$ over L, and if z is not in $\mathbf{C}$ then we can find an elementary symmetric function w in L also not in $\mathbf{C}$. Now (3.1) follows from (3.2) using $h(w) \ge 1$ and $h(z) = h(z_1) = \cdots = h(z_m)$ (see [14] p.52).

We next proceed to verify that

$$h(x_i) \le 1 \ (i = 1, \ldots, n). \tag{3.3}$$

Let y be any other generic linear polynomial in $x_1, \ldots, x_n$. Through the Chow form it is well known that $y^\Delta + w_1 y^{\Delta-1} + \cdots + w_\Delta = 0$, where w_j in $\mathbf{C}[y_1, \ldots, y_r]$ has total degree at most j $(j = 1, \ldots, \Delta)$. Thus for any ultrametric valuation there is j with

$$|y|^\Delta \le \max\{|w_1 y^{\Delta-1}|, \ldots, |w_\Delta|\} = |w_j y^{\Delta-j}| \le E^j |y|^{\Delta-j}, \qquad \blacksquare$$

where $E = e$ if the valuation extends the total degree valuation and otherwise $E = 1$. It follows that $\max\{1, |y|\} \le E$. Now each x_i $(i = 1, \ldots, n)$ is a linear combination of such y with coefficients in $\mathbf{C}$, and so we get the same upper bound for $\max\{1, |x_i|\}$. The estimate (3.3) follows at once.

We are going to apply the "multiplicative dependence estimate" Lemma 2.2 of [5] (p.457) to the $x_1, \ldots, x_n$. As $\mathcal{X}$ is anomalous, these generate a subgroup of rank at most $n-1$ over Z. Using (3.3) we get for any positive integer T a non-zero $(a_1, \ldots, a_n)$ in $\mathbf{Z}^n$ with $|a_i| \le T$ $(i = 1, \ldots, n)$ and $h(z) \le cT^{-\frac{1}{n-1}}$ for $z = x^{\mathbf{a}}$, where c depends only on n. Choosing T minimally to contradict (3.1), we deduce that the resulting z is constant on $\mathcal{X}$. This proves the present lemma.

From the discussion after the proof of Lemma 2.4 we see that the lemma can be regarded as an effective supplement to Chabauty's Theorem for $\mathbf{G}_m^n$ (there is an alternative proof using $\mathcal{X}, \mathcal{X}\mathcal{X}, \mathcal{X}\mathcal{X}\mathcal{X}, \ldots$). An example of familiar type shows in fact that the dependence on the degree Δ is best possible, at least for curves. To see this, let $t_1, \ldots, t_{n-1}$ be generic linear polynomials in a single variable t, and consider the curve $\mathcal{X}$ parametrized by

$$x_1 = t_1^b,\ x_2 = t_1 t_2^b,\ x_3 = t_2 t_3^b,\ \ldots,\ x_{n-1} = t_{n-2} t_{n-1}^b,\ x_n = t_{n-1}.$$

It has degree $\Delta \leq b+1$. And if $x^{\mathbf{a}}$ is constant on $\mathcal{X}$, then we get the equations

$$ba_1 + a_2 = 0,\ ba_2 + a_3 = 0,\ \ldots,\ ba_{n-1} + a_n = 0$$

for $\mathbf{a} = (a_1, \ldots, a_n)$. Thus

$$|\mathbf{a}| \ \geq\ |a_n| \ =\ b^{n-1}|a_1| \ \geq\ b^{n-1} \ \geq\ (\Delta - 1)^{n-1}$$

if $|\mathbf{a}| \neq 0$.

Proof of Proposition 3.3. We will use induction on the dimension r. □

The case $r = 1$ of curves is easy, for then $\mathcal{Y} = \mathcal{X}$ and we can appeal to Lemma 3.2 to get $\mu(n, 1) = n - 1$.

Therefore we take $\mathcal{X}$ with dimension $r \geq 2$ and we can assume that the Proposition 3.1 holds for anomalous subvarieties of varieties with lower dimension. We are going to verify it for any $\mathcal{X}$-anomalous subvariety $\mathcal{Y}$ in $\mathcal{X}$. For this purpose we may assume that $\mathcal{Y}$ is maximal, of dimension say s. Again from Lemma 3.2 we can suppose $\mathcal{Y} \neq \mathcal{X}$ so $s \leq r - 1$.

By definition $\mathcal{Y}$ lies in some translate of some subgroup $H(\mathbf{a}_1, \ldots, \mathbf{a}_h)$ with $\mathbf{a}_1, \ldots, \mathbf{a}_h$ $\mathbf{Q}$-linearly independent in $\mathbf{Z}^n$. So (1.1) gives $h \geq r - s + 1 \geq 2$; and furthermore, by enlarging the subgroup if necessary, we can assume $h = r - s + 1$.

Lemma 2.2 now implies that $\mathrm{rank}_{\mathcal{Y}} J(\mathbf{a}_1, \ldots, \mathbf{a}_h; \mathcal{X}) \leq n - r + h - 1$; that is, many equations vanish on $\mathcal{Y}$.

As for $\mathcal{X}$, suppose first that $\mathrm{rank}_{\mathcal{X}} J(\mathbf{a}_1, \ldots, \mathbf{a}_h; \mathcal{X}) > n - r + h - 1$. This will enable us to find one of these equations that does not vanish on $\mathcal{X}$. Namely, any minor F of size at least $n-r+h$ that does not vanish identically on $\mathcal{X}$. Then $F = 0$ intersects $\mathcal{X}$ in a variety of dimensions at most $r - 1$, and $\mathcal{Y}$ lies in some component $\mathcal{X}'$ of this intersection. It is clear from (1.1) that $\mathcal{Y}$ is $\mathcal{X}'$-anomalous. So we can use our induction hypothesis to find some non-trivial $x^{\mathbf{a}}$ constant on $\mathcal{Y}$. The polynomial $x_1 \ldots x_n F$ has degree at most $c\Delta$ with c depending only on n, and so by Bézout the degree of $\mathcal{X}'$ is at most $c\Delta^2$ with a similar c. Thus the bound for $|\mathbf{a}|$ has the required shape with $\mu(n, r) = 2\mu(n, r - 1)$.

It remains to consider the case $\mathrm{rank}_{\mathcal{X}} J(\mathbf{a}_1, \ldots, \mathbf{a}_h; \mathcal{X}) \ \leq\ n - r + h - 1$. Then by Lemma 2.1 we see that $x^{\mathbf{a}_1}, \ldots, x^{\mathbf{a}_h}$ are algebraically dependent on $\mathcal{X}$. With $\varphi = \varphi(\mathbf{a}_1, \ldots, \mathbf{a}_h)$, this means that the image $\varphi(\mathcal{X})$ in $\mathbf{G}_m^h$ has dimension at most $h - 1$.

And since $\mathcal{Y}$ was maximal, we see from Lemma 2.3 that this dimension is exactly $h - 1$. The closure of $\varphi(\mathcal{X})$ in $\mathbf{G}_m^h$ is therefore a hypersurface $\mathcal{W}$. By FDT(b) we can find an open dense subset U of $\mathcal{W}$ such that every component $\mathcal{Z}$ of $\mathcal{X}\cap\varphi^{-1}(w)$ for every w in U has dimension $r-(h-1)=s$. Furthermore, by shrinking if necessary, we can also suppose that every point of U is non-singular on $\mathcal{W}$.

Now each $\mathcal{Z}$ above is $\mathcal{X}$-anomalous, because it too lies in a translate of $H(\mathbf{a}_1,\ldots,\mathbf{a}_h)$. Provided $\mathcal{Z}$ does not lie in the singular locus of $\mathcal{X}$, we are therefore in a position to apply Lemma 2.5 to $\mathcal{Z}$. We obtain $\mathbf{z}_\mathcal{Z}\neq 0$ in $\mathbf{C}^n$ such that $\mathrm{rank}_\mathcal{Z} J(\mathbf{z}_\mathcal{Z};\mathcal{X})\leq n-r$. These give equations vanishing on $\mathcal{Z}$; and, exactly as above for $\mathcal{Y}$, we proceed to find one of these not vanishing on $\mathcal{X}$.

First we claim that $\mathrm{rank}_\mathcal{X} J(\mathbf{z}_\mathcal{Z};\mathcal{X})>n-r$. For otherwise Lemma 2.4 would imply that $\mathcal{X}$ was $\mathcal{X}$-anomalous. Since $\mathcal{Y}$ was maximal, this would mean that $\mathcal{Y}=\mathcal{X}$; but we excluded this case above.

As above for $\mathcal{Y}$, we can find a minor G not vanishing identically on $\mathcal{X}$, and so we can use the induction hypothesis to find non-zero $\mathbf{a}_\mathcal{Z}$ in $\mathbf{Z}^n$, such that $x^{\mathbf{a}_\mathcal{Z}}$ is constant on $\mathcal{Z}$, with $|\mathbf{a}_\mathcal{Z}|\leq c\Delta^{2\mu}$ for $\mu=\mu(n,r-1)$ and c depending only on n.

What if $\mathcal{Z}$ does lie in the singular locus of $\mathcal{X}$? This locus is defined in $\mathcal{X}$ by the vanishing of polynomials of total degree at most $c\Delta$ for some c also depending only on n, and now we can use one of these in place of G to find $\mathbf{a}_\mathcal{Z}$ with the properties just above.

Now the union of all such $\mathcal{Z}$ is $\mathcal{X}\cap\varphi^{-1}(U)$, which is dense in $\mathcal{X}$. By the Box Principle applied to the finitely many possibilities for $\mathbf{a}_\mathcal{Z}$, we can therefore find a dense subset T of $\mathcal{X}\cap\varphi^{-1}(U)$ together with non-zero $\mathbf{a}$ in $\mathbf{Z}^n$, such that $x^\mathbf{a}$ is constant on T, also with $|\mathbf{a}|\leq c\Delta^{2\mu}$.

We now claim that $x^\mathbf{a}$ is algebraically dependent on $x^{\mathbf{a}_1},\ldots,x^{\mathbf{a}_h}$ on $\mathcal{X}$; in other words, that $x^{\mathbf{a}_1},\ldots,x^{\mathbf{a}_h}$ together with $x^\mathbf{a}$ have the same transcendence degree $h-1$ on $\mathcal{X}$ as $x^{\mathbf{a}_1},\ldots,x^{\mathbf{a}_h}$. We can suppose that $x^{\mathbf{a}_1},\ldots,x^{\mathbf{a}_{h-1}}$ are algebraically independent on $\mathcal{X}$. It will therefore suffice to check that $x^{\mathbf{a}_1},\ldots,x^{\mathbf{a}_{h-1}},x^\mathbf{a}$ are algebraically dependent on $\mathcal{X}$. But if they were independent, then Lemma 2.1 would show that $\mathrm{rank}_\mathcal{X} J(\mathbf{a}_1,\ldots,\mathbf{a}_{h-1},\mathbf{a};\mathcal{X})>n-r+h-1$. So there would exist a minor G of size at least $n-r+h$ that does not vanish identically on $\mathcal{X}$. As T is dense in $\mathcal{X}$, we could then find t in T with $G(t)\neq 0$. However, t lies on one of the components $\mathcal{Z}$ above, which as we have seen are all $\mathcal{X}$-anomalous of dimension s. It lies in a translate of not only $H(\mathbf{a}_1,\ldots,\mathbf{a}_h)$ but also $H(\mathbf{a}_1,\ldots,\mathbf{a}_{h-1},\mathbf{a})$. Therefore Lemma 2.2 shows that $\mathrm{rank}_\mathcal{Z} J(\mathbf{a}_1,\ldots,\mathbf{a}_{h-1},\mathbf{a};\mathcal{X})\leq n-r+h-1$. And so the above minor G vanishes on $\mathcal{Z}$ and so at t; a contradiction. This contradiction establishes the current claim.

We next proceed to deduce that $x^{\mathbf{a}}$ is constant on the original $\mathcal{Y}$; and this will complete the induction step and thereby the proof of the Proposition 3.1, still with $\mu(n,r) = 2\mu(n,r-1)$.

Now $\varphi = \varphi(\mathbf{a}_1,\dots,\mathbf{a}_h)$ is constant on each translate of $H(\mathbf{a}_1,\dots,\mathbf{a}_h)$ and so on $\mathcal{Y}$; call this constant point w, so that $\mathcal{Y}$ lies in $\mathcal{X}\cap\varphi^{-1}(w)$. In fact, $\mathcal{Y}$ must be a component of $\mathcal{X}\cap\varphi^{-1}(w)$, for if not, it would lie in a component of dimension strictly bigger than s. By (1.1) this larger component would still be $\mathcal{X}$-anomalous, contradicting maximality.

Thus $\mathcal{X}\cap\varphi^{-1}(w) = \mathcal{Y}\cup\mathcal{X}_0$ for some subvariety $\mathcal{X}_0$ (possibly empty) of $\mathcal{X}$ not containing $\mathcal{Y}$. Choose y in $\mathcal{Y}\setminus\mathcal{X}_0$ and define $\lambda = x^{\mathbf{a}}(y)$. We have a map $\psi = \varphi(\mathbf{a}_1,\dots,\mathbf{a}_h,\mathbf{a})$ from $\mathbf{G}_m^n$ to $\mathbf{G}_m^{h+1}$, and by the definition of λ, the intersection $\mathcal{X}\cap\psi^{-1}(w,\lambda)$ contains y. Let $\mathcal{Y}_?$ be a component of $\mathcal{X}\cap\psi^{-1}(w,\lambda)$ containing y. As $\mathcal{Y}_?$ lies in $\mathcal{X}\cap\varphi^{-1}(w) = \mathcal{Y}\cup\mathcal{X}_0$, it follows by the choice of y that $\mathcal{Y}_?$ lies in $\mathcal{Y}$.

On the other hand, the claim just above, together with the algebraic dependence of $x^{\mathbf{a}_1},\dots,x^{\mathbf{a}_h}$, implies that $\psi(\mathcal{X})$ has dimension at most $h-1$. So FDT(a) shows that $\mathcal{Y}_?$ has dimension at least $r-(h-1) = s$. Because this is also the dimension of $\mathcal{Y}$, we conclude that $\mathcal{Y}_? = \mathcal{Y}$. Finally $x^{\mathbf{a}} = \lambda$ on $\mathcal{Y}_?$ and therefore also on $\mathcal{Y}$.

As mentioned above, this completes the proof of the Proposition 3.1.

We can now establish Theorem 1.4.

Part (a), holding for any fixed torus H, is a relatively easy consequence of the Fiber Dimension Theorem. Applying the automorphism α_H discussed in section 1, we can suppose that $H = \{1\}^h\times\mathbf{G}_m^{n-h}$. The effect of this is to reduce α_H to the identity.

Let π_h be the projection to $\mathbf{G}_m^{n-h}$ also as in section 1. We temporarily define $\mathcal{Z}_?$ as the set of all x in $\mathcal{X}$ with $\dim_x(\mathcal{X}\cap\pi_h^{-1}(\pi_h(x)))\ge s$. We start by verifying that $\mathcal{Z}_? = \mathcal{Z}_H$.

Certainly, if x is in $\mathcal{Z}_H$, then x lies in a subvariety $\mathcal{Y}$ of $\mathcal{X}$ which also lies in a coset of H; and by (1.4) the dimension of $\mathcal{Y}$ is $r-h+1 = s$. But $\pi_h^{-1}(\pi_h(x))$ is the unique translate of H containing x. So its dimension at x is at least that of $\mathcal{Y}$, and therefore x lies in $\mathcal{Z}_?$. Conversely, if x is in $\mathcal{Z}_?$ then the component $\mathcal{Y}_x$ of $\mathcal{X}\cap\pi_h^{-1}(\pi_h(x))$ through x has dimension at least s. If $\dim\mathcal{Y}_x = s$, then $\mathcal{Y}_x$ is as in (1.4) so $\mathcal{Y}_x$ lies in $\mathcal{Z}_H$ and therefore x too lies in $\mathcal{Z}_H$. If $\dim\mathcal{Y}_x > s$, let us take any irreducible subvariety $\mathcal{Y}'_x$ through x of dimension s; this will necessarily be anomalous. Now $\mathcal{Y}'_x$ is as in (1.4) so we conclude just as above that x lies in $\mathcal{Z}_H$. This establishes $\mathcal{Z}_? = \mathcal{Z}_H$.

Suppose first that $x_1,\dots,x_h$ are algebraically dependent on $\mathcal{X}$. Then $\pi_h(\mathcal{X})$ has dimension at most $h-1$. So by FDT(a) the set $\mathcal{Y}_x$ above has dimension at least $r-(h-1) = s$ for every x in $\mathcal{X}$. Therefore $\mathcal{Z}_H = \mathcal{Z}_? = \mathcal{X}$ (and in particular $\mathcal{X}^{oa}$ is empty). Now $U_H = \pi_h(\mathcal{Z}_H) = \pi_h(\mathcal{X})$ is clearly not dense in $\mathbf{G}_m^h$. So (a) holds in this case.

What happens if $x_1,\dots,x_h$ are algebraically independent on $\mathcal{X}$? Now by FDT(b) there is an open dense set in $\mathbf{G}_m^h$ consisting of points w with $\dim(\mathcal{X}\cap\pi_h^{-1}(w)) = r-h = s-1$. So, $U_H = \pi_h(\mathcal{Z}_H) = \pi_h(\mathcal{Z}_?)$ cannot be dense in $\mathbf{G}_m^h$, because all its points w satisfy $\dim(\mathcal{X}\cap\pi_h^{-1}(w)) \geq s$. Therefore $\mathcal{Z}_H \neq \mathcal{X}$, otherwise U_H would be $\pi_h(\mathcal{X})$ which is now dense in $\mathbf{G}_m^h$. And by FDT(c) the set $\mathcal{Z}_H = \mathcal{Z}_?$ is still closed in $\mathcal{X}$. So (a) holds in this case too; and this completes the proof of Theorem 1(a).

For part (b) we establish the existence of the finite collection Φ using induction on the dimension $r = \dim\mathcal{X}$. In fact, it is necessary here to load the induction with an extra condition on degrees. Namely,

$$\deg H \;\leq\; c(n,r)(\deg\mathcal{X})^{\kappa(n,r)} \tag{3.4}$$

for every H in Φ. This gives the Uniform Structure Theorem as described in the Introduction, because it is well known that any torus H can be defined by equations $x^{\mathbf{a}} = 1$ with $|\mathbf{a}| \leq c(n)(\deg H)^{\lambda(n)}$, even for $\lambda(n) = 1$. See for example [24] (p.521) or [3] (p.89).

If $r = 1$ then $\mathcal{X}$ is a curve, and the only candidate for any anomalous subvariety, maximal or not, is $\mathcal{X}$ itself, which is possible only if $\mathcal{X}$ lies in a proper coset gH. When this is so, then by Lemma 3.2 we can suppose, indeed, that H belongs to a finite collection satisfying (3.4). Now $\mathcal{Z}_H$ contains $\mathcal{X}$ (of course $\mathcal{Z}_H = \mathcal{X}$) and so $U_H = \pi_h(\alpha_H(\mathcal{Z}_H))$ contains $\pi_h(\alpha_H(\mathcal{X}))$. This latter set lies in $\pi_h(\alpha_H(gH))$, which reduces to the single point $\pi_h(\alpha_H(g))$. So U_H is this point. Thus $U_H \times \mathbf{G}_m^{n-h} = \alpha_H(H\mathcal{Z}_H)$ contains $\alpha_H(g)$, and g lies in $H\mathcal{Z}_H$. So by adjusting g we can take it in $\mathcal{Z}_H$ as claimed. Here, of course, $\mathcal{X}^{oa}$ is empty.

Next assume $r \geq 2$, and let $\mathcal{Y}$ be a maximal anomalous subvariety of $\mathcal{X}$ of dimension s. By the Proposition 3.1, there is some non-trivial relation $x^{\mathbf{a}} = \lambda$ holding on $\mathcal{Y}$, with $|\mathbf{a}| \leq c(n,r)(\deg\mathcal{X})^{\mu(n,r)}$. We can assume that $\mathbf{a}$ is primitive and then find $\mathbf{a}_1,\dots,\mathbf{a}_{n-1}$, satisfying similar bounds, such that $\alpha = \varphi(\mathbf{a}_1,\dots,\mathbf{a}_{n-1},\mathbf{a})$, in the notation of section 2, is an automorphism. Now $x_n = \lambda$ on $\alpha(\mathcal{X})$. Further it is easy to show that $\deg\alpha(\mathcal{X}) \leq c(\deg\mathcal{X})^{\nu}$ for c,ν depending only on n. Thus from the point of view of (3.4) we may assume that α is the identity and $\mathbf{a} = (0,\dots,0,1)$.

If $x_n = \lambda$ on $\mathcal{X}$, then $\mathcal{X}$ is anomalous in $\mathcal{X}$, so maximality implies $\mathcal{Y} = \mathcal{X}$ and we can finish as in the curve case just above.

Otherwise, $x_n = \lambda$ intersects $\mathcal{X}$ in a subvariety $\tilde{\mathcal{X}}$ of dimension $r-1$. Let $\mathcal{Y}'$ be the projection of $\mathcal{Y}$ to $\mathbf{G}_m^{n-1}$, still of dimension s. It lies in the projection $\mathcal{X}'$, also of dimension $r-1$, to $\mathbf{G}_m^{n-1}$ of some component of $\tilde{\mathcal{X}}$. Also $\deg\mathcal{X}' \leq \deg\mathcal{X}$. Now $\mathcal{Y}'$ is anomalous in $\mathcal{X}'$. For, by (1.1) $\mathcal{Y}$ lies in a coset in $\mathbf{G}_m^n$ of dimension at most $n-r+s-1$. This coset projects to one of dimension at most $n-r+s-1 = (n-1)-(r-1)+\dim \mathcal{Y}'-1$ in $\mathbf{G}_m^{n-1}$ which contains $\mathcal{Y}'$, and so, indeed, we see that $\mathcal{Y}'$ is anomalous in $\mathcal{X}'$.

In fact $\mathcal{Y}'$ is maximal anomalous in $\mathcal{X}'$. For

$$\mathcal{Y}' \subseteq \mathcal{Y}'_? \subseteq \mathcal{X}' \subseteq \mathbf{G}_m^{n-1}$$

with $\mathcal{Y}'_?$ anomalous in $\mathcal{X}'$ implies

$$\mathcal{Y} = \mathcal{Y}' \times \{\lambda\} \subseteq \mathcal{Y}'_? \times \{\lambda\} \subseteq \mathcal{X}' \times \{\lambda\}$$

and the extreme right-hand side here is contained in $\tilde{\mathcal{X}}$. Now $\mathcal{Y}'_? \times \{\lambda\}$ is anomalous in $\mathcal{X}$ because it is contained in a coset of dimension $(n-1)-(r-1)+\dim \mathcal{Y}'_? - 1 = n-r+\dim(\mathcal{Y}'_? \times \{\lambda\})-1$. Thus the maximality of $\mathcal{Y}$ implies $\mathcal{Y}' \times \{\lambda\} = \mathcal{Y}'_? \times \{\lambda\}$, thus $\mathcal{Y}' = \mathcal{Y}'_?$; and so, indeed, $\mathcal{Y}'$ is maximal anomalous in $\mathcal{X}'$.

Therefore, by our induction hypothesis, $\mathcal{Y}'$ lies in a translate of an algebraic subgroup H' of $\mathbf{G}_m^{n-1}$ of dimension $(n-1)-(r-1)+s-1 = n-r+s-1$ with $\deg H' \le c(n-1,r-1)(\deg \mathcal{X}')^{\kappa(n-1,r-1)}$. And finally, $\mathcal{Y} = \mathcal{Y}' \times \{\lambda\}$ lies in a similar translate gH in $\mathbf{G}_m^n$ with $H = H' \times \{1\}$: of course with a similar bound for $\deg H$. And $\mathcal{Y}$ must be a component of $\mathcal{X} \cap gH$, because otherwise the component containing $\mathcal{Y}$ would contradict the maximality of $\mathcal{Y}$. Now $\mathcal{Z}_H$ contains $\mathcal{Y}$ so $U_H = \pi_h(\alpha_H(\mathcal{Z}_H))$ contains $\pi_h(\alpha_H(\mathcal{Y}))$. The latter set lies in $\pi_h(\alpha_H(gH)) = \pi_h(\alpha_H(g))$, so must be this point. Thus U_H contains this point; and we conclude that g can be taken in $\mathcal{Z}_H$ just as in the curve case above.

And finally, it is clear that $\mathcal{X} \setminus \mathcal{X}^{oa}$ is the union of the $\mathcal{Z}_H$, because it suffices to remove the maximal anomalous subvarieties. This completes the proof of the Uniform Structure Theorem.

Using the effective version of the Fiber Dimension Theorem mentioned in section 2, we can easily check that the $\deg \mathcal{Z}_H$ are bounded as in (3.4). We omit the details.

4 Proof of Theorems 1.4 and 1.5

We begin by remarking that the Structure Theorem, although here proved only over the complex field $\mathbf{C}$, remains valid over any algebraically closed field of zero characteristic. In particular, if $\mathcal{X}$ is defined over $\overline{\mathbf{Q}}$, then the sets $\mathcal{Z}_H$ are also defined over $\overline{\mathbf{Q}}$.

We start with Theorem 1.5. Clearly this is a special case of the following result.

Theorem 4.1. For $n \geq 2$ let $\mathcal{X}$ be an irreducible variety in $\mathbf{G}_m^n$, of dimension at most $n-2$, defined over $\overline{\mathbf{Q}}$. Then there exists $B = B(\mathcal{X})$ depending only on $\mathcal{X}$ with the following property. Suppose $\zeta_1, \dots, \zeta_n$ are roots of unity, $a_1, \dots, a_n$ are rational integers, and τ is a non-zero complex number with $(\zeta_1\tau^{a_1}, \dots, \zeta_n\tau^{a_n})$ in $\mathcal{X}$. Then there exist rational integers $b_1, \dots, b_n$ with

$$0 < \max\{|b_1|, \dots, |b_n|\} \le B, \; (\zeta_1\tau^{a_1})^{b_1} \dots (\zeta_n\tau^{a_n})^{b_n} = 1. \qquad \square$$

Proof. We observe at once that the hypotheses and conclusion of the theorem are unaffected by applying an automorphism of $\mathbf{G}_m^n$. Therefore for notational simplicity we shall freely use such automorphisms without changing the symbols $\zeta_1, \dots, \zeta_n, a_1, \dots, a_n$ and τ. Replacing τ by some integral power, we can easily see that no loss of generality is involved in supposing $a_1, \dots, a_n$ to be coprime. We will now use induction on n. ■

The case $n = 2$ is easy, for then $(\zeta_1\tau^{a_1}, \zeta_2\tau^{a_2})$ is to a fixed algebraic point with multiplicatively dependent coordinates. So there are bounded b_1, b_2 in $\mathbf{Z}$, not both zero, with $(\zeta_1\tau^{a_1})^{b_1}(\zeta_2\tau^{a_2})^{b_2} = 1$. A similar argument works in any dimension in case $x = (\zeta_1\tau^{a_1}, \dots, \zeta_n\tau^{a_n})$ belongs to a fixed set of algebraic points.

We next suppose that $n \ge 3$. We deal first with the possibility that x does not lie in $\mathcal{X}^o$. From the results of [8] quoted in section 1, it follows that x lies in some translate gH of a positive-dimensional torus H belonging to a finite collection, with gH itself in $\mathcal{X}$. After an automorphism we can suppose that $H = \{1\}^h \times \mathbf{G}_m^{n-h}$ for some $h \le n-1$. It follows that g lies in $\mathcal{V} \times \mathbf{G}_m^{n-h}$, itself in $\mathcal{X}$, for some fixed subvariety $\mathcal{V}$ in $\mathbf{G}_m^h$ also defined over $\overline{\mathbf{Q}}$. Projecting down to $\mathbf{G}_m^h$, we obtain from x a point $v = (\zeta_1\tau^{a_1}, \dots, \zeta_h\tau^{a_h})$ in $\mathcal{V}$. Now

$$\dim \mathcal{V} = \dim(\mathcal{V} \times \mathbf{G}_m^{n-h}) - (n-h) \le \dim \mathcal{X} - (n-h) \le n-2-(n-h) = h-2.$$

So our induction hypothesis applies. It tells us that there exist rational integers $b_1, \dots, b_h$ with

$$0 < \max\{|b_1|, \dots, |b_h|\} \le B(\mathcal{V}), \; (\zeta_1\tau^{a_1})^{b_1} \dots (\zeta_h\tau^{a_h})^{b_h} = 1.$$

So we get the required conclusion for x not in $\mathcal{X}^o$, with $b_{h+1} = \dots = b_n = 0$.

From now on we shall assume that x lies in $\mathcal{X}^o$. Put $\mathbf{a} = (a_1, \dots, a_n)$ in $\mathbf{Z}^n$. The Geometry of Numbers (or Siegel's Lemma) gives $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ in $\mathbf{Z}^n$, perpendicular to $\mathbf{a}$, linearly independent with $|\mathbf{u}_1| \dots |\mathbf{u}_{n-1}| \le c|\mathbf{a}|$ for c depending only on n. Henceforth we shall use this same symbol c for similar possibly different constants. We can assume $|\mathbf{u}_1| \le \dots \le |\mathbf{u}_{n-1}|$. Let k be a number field containing a field of definition for $\mathcal{X}$ as well as $\zeta_1, \ldots, \zeta_n$. We can find roots of unity $\eta_1, \ldots, \eta_{n-2}$ in k such that the two-dimensional torsion coset T_2 defined by $x^{\mathbf{u}_j} = \eta_j$ ($j = 1, \ldots, n-2$) contains the one-dimensional torsion coset T_1 parametrized by $x_i = \zeta_i t^{a_i}$ ($i = 1, \ldots, n$), simply by substituting the latter equations into $x^{\mathbf{u}_j}$. The degree of T_2 is then at most $c|\mathbf{u}_1| \ldots |\mathbf{u}_{n-2}|$ so at most $c|\mathbf{a}|^{\frac{n-2}{n-1}}$. As x lies in T_1, it also lies in $\mathcal{X} \cap T_2$.

Suppose first that x is an isolated point in $\mathcal{X} \cap T_2$. By Bézout the degree d of x over k satisfies

$$d \ \leq \ C|\mathbf{a}|^{\frac{n-2}{n-1}}, \tag{4.1}$$

where now C depends also on the degree of a field of definition for $\mathcal{X}$. Henceforth we shall use this same symbol C for possibly different constants depending only on $\mathcal{X}$. Because $a_1, \ldots, a_n$ are coprime, we see that in fact $d = [k(\tau) : k]$.

If τ is itself a root of unity, then x is a torsion point on $\mathcal{X}$. As x lies in $\mathcal{X}^o$, the Corollary of [8] (p.336) shows that x must belong to a fixed finite set, and by the above remark we are again done.

Otherwise, if τ is not a root of unity, then Theorem 1.1 of Amoroso and Zannier [2] (p.712) implies that for any $\epsilon > 0$ there is $C(\epsilon) > 0$, depending only on ϵ and a field of definition for $\mathcal{X}$, such that the absolute logarithmic height satisfies

$$h(\tau) \ \geq \ \frac{1}{C(\epsilon) d^{1+\epsilon}}. \tag{4.2}$$

But since x is in $\mathcal{X}^o$, the result from [24] mentioned in section 1 gives $h(x) \leq C$. Also

$$h(x) \ = \ h(\zeta_1 \tau^{a_1}) + \cdots + h(\zeta_n \tau^{a_n}) \ \geq \ |\mathbf{a}| h(\tau).$$

Comparing these upper bounds with the lower bound (4.2), taking (4.1) into account and fixing ϵ small enough as a function of n, we find that $|\mathbf{a}| \leq C$.

Because $a_1, \ldots, a_n$ are coprime, we can now use a bounded automorphism to ensure that $\mathbf{a} = (0, \ldots, 0, 1)$. And if necessary changing τ, we can assume $\zeta_n = 1$, so that now $x = (\zeta_1, \ldots, \zeta_{n-1}, \tau)$. Projecting down to $\mathbf{G}_m^{n-1}$, we obtain a torsion point $x' = (\zeta_1, \ldots, \zeta_{n-1})$ in the projection $\mathcal{X}'$ of $\mathcal{X}$. By the known results on torsion points on varieties mentioned in section 1, we can assume that x' lies in a fixed algebraic subgroup H' itself contained in $\mathcal{X}'$. Now

$$\dim H' \ \leq \ \dim \mathcal{X}' \ \leq \ \dim \mathcal{X} \ \leq \ n-2.$$

So there exist rational integers $b_1, \dots, b_{n-1}$ with

$$0 < \max\{|b_1|, \dots, |b_{n-1}|\} \le B_0(\mathcal{X}'), \ \zeta_1^{b_1} \dots \zeta_{n-1}^{b_{n-1}} = 1;$$

and we get the required conclusion for x an isolated point in $\mathcal{X} \cap T_2$.

It remains to consider the possibility that x lies in a positive-dimensional component $\mathcal{Y}$ of $\mathcal{X} \cap T_2$. Then $\mathcal{Y}$ is anomalous in $\mathcal{X}$. It is contained in a maximal anomalous subvariety $\mathcal{Y}_m$, say of dimension $s \le \dim \mathcal{X}$. From Theorem 1.4 we can assume that $\mathcal{Y}_m$ lies in some translate gH of a fixed torus H with dimension $n - \dim \mathcal{X} + s - 1 \le n - 1$. The intersection $K = T_2 \cap gH$ contains $\mathcal{Y}$ and therefore has dimension 1 or 2. It is a coset itself.

Suppose first that $\dim K = 1$. Then $\mathcal{Y}$ is a component of K, so $\mathcal{Y}$ is something removed to give $\mathcal{X}^o$. As x lies in $\mathcal{Y}$, this contradicts the assumption that x lies in $\mathcal{X}^o$.

Thus $\dim K = 2$. This means that some component of T_2 lies in gH, and it follows that $gH = g_0H$ for any torsion point g_0 in this component. Again we can assume $H = \{1\}^h \times \mathbf{G}_m^{n-h}$ for some $h \le n-1$, which amounts to taking α_H as the identity. Now by Theorem 1.4(b) we can take g in $\mathcal{Z}_H$ and so $\pi_h(g)$ lies in $\pi_h(\mathcal{Z}_H) = U_H$. But here $\pi_h(g) = \pi_h(g_0)$ is a torsion point, also identical with $\pi_h(x) = v$ above. Again using the known results on torsion points on varieties, we can assume that v lies in a fixed algebraic subgroup H' itself contained in $\mathcal{V}$ the closure of U_H in $\mathbf{G}_m^h$. From Theorem 1.4(a) and the discussion in section 1 we know that $\mathcal{V}$ is not $\mathbf{G}_m^h$, and so $\dim H' < h$. Hence there exist rational integers $b_1, \dots, b_h$ with

$$0 < \max\{|b_1|, \dots, |b_h|\} \le B_0(\mathcal{V}), \ (\zeta_1 \tau^{a_1})^{b_1} \dots (\zeta_h \tau^{a_h})^{b_h} = 1.$$

So again we get the required conclusion for x. The proof of Theorem 1.6, so also that of Theorem 1.5, is finally complete.

We next deduce Theorem 1.4, again by induction on n.

If $\dim \mathcal{X} \ge n-1$, then we can take Ψ to consist only of $\mathbf{G}_m^n$, and so the result is trivial. This also does the starting case $n = 1$. So we can assume $n \ge 2$ and $\dim \mathcal{X} \le n-2$.

Let x be any point of $\mathcal{X} \cap \mathcal{H}_1$. Then of course x has the shape in Theorem 1.6, and so there exist rational integers $b_1, \dots, b_n$ as in Theorem 1.6. This means that x lies in the essentially fixed algebraic subgroup defined by $x_1^{b_1} \dots x_n^{b_n} = 1$. So it lies in some component T_{n-1}, which after the usual automorphism we can assume to be defined by $x_n = \zeta_n$ for a fixed root of unity ζ_n.

Now the component $\tilde{\mathcal{X}}$ of $\mathcal{X} \cap T_{n-1}$ through x has the form $\tilde{\mathcal{X}} = \mathcal{X}' \times \{\zeta_n\}$ for $\mathcal{X}'$ in $\mathbf{G}_m^{n-1}$. And likewise, $x = x' \times \{\zeta_n\}$ for x' in $\mathbf{G}_m^{n-1}$. As x' lies in $\mathcal{X}' \cap \mathcal{H}_1$, we can use the induction hypothesis to see that x' lies in one of a finite number of fixed translates T' in $\mathbf{G}_m^{n-1}$ of tori by torsion points, with $\dim(\mathcal{X}' \cap T') \geq \dim T' - 1$. Thus x lies in $T = T' \times \{\zeta_n\}$ itself in T_{n-1}; and now $\mathcal{X} \cap T = (\mathcal{X} \cap T_{n-1}) \cap T$ contains $\tilde{\mathcal{X}} \cap T = (\mathcal{X}' \cap T') \times \{\zeta_n\}$. Thus

$$\dim(\mathcal{X} \cap T) \;\geq\; \dim(\mathcal{X}' \cap T') \;\geq\; \dim T' - 1 \;=\; \dim T - 1.$$

We now see our collection $\Psi = \Psi_{\mathcal{X}}$, and since x was arbitrary we have shown that $\mathcal{X} \cap \mathcal{H}_1$ lies in the union over Ψ of the $(\mathcal{X} \cap T) \cap \mathcal{H}_1$. So clearly $\mathcal{X} \cap \mathcal{H}_1$ is this union, and Theorem 1.4 is proved.

5 Conjectures and proof of Theorem 1.7

In fact the Bounded Height Conjecture may not be absolutely sharp. Again write $r = \dim \mathcal{X}$, and let $\mathcal{Y}$ be an anomalous subvariety, of dimension s with $1 \leq s \leq r$, of $\mathcal{X}$ inside a coset K, of dimension $n - h < n - r + s$, as in (1.1). It may happen that K is contained in an algebraic subgroup H of dimension at most $n - r + s$ (for example if $\mathcal{Y} = \mathcal{X}$ so $s = r$). In this case one may show in the following way that the points of $\mathcal{Y} \cap \mathcal{H}_{n-r}$ in $\mathcal{X} \cap \mathcal{H}_{n-r}$ definitely do not have bounded height. After an automorphism we can assume that the coordinates $x_1, \ldots, x_h$ are constants $\xi_1, \ldots, \xi_h$ on K; and the special condition on K means that these constants have multiplicative rank at most $h - r + s \leq h$. So after another automorphism we can assume that $\xi_1 = \zeta_1, \ldots, \xi_{r-s} = \zeta_{r-s}$ are roots of unity. We can also assume that $x_{h+1}, \ldots, x_{h+s}$ are algebraically independent on $\mathcal{Y}$ (note $h + s \leq n$). If one of $\xi_{r-s+1}, \ldots, \xi_h$, say ξ, is not a root of unity then we can intersect $\mathcal{Y}$ with $x_{h+1} = \xi^{b_{h+1}}, \ldots, x_{h+s} = \xi^{b_{h+s}}$, and in general we get points of $\mathcal{Y}$ in $\mathcal{X}$ of the shape

$$(\zeta_1, \ldots, \zeta_{r-s}, \xi_{r-s+1}, \ldots, \xi_h, \xi^{b_{h+1}}, \ldots, \xi^{b_{h+s}}, x_{h+s+1}, \ldots, x_n).$$

The multiplicative rank is therefore at most $(h - r + s) + (n - h - s) = n - r$, and so they lie in $\mathcal{H}_{n-r}$. And as, say, $b_{h+1} \to \infty$ we get points with unbounded height. The argument is simpler if $\xi_{r-s+1}, \ldots, \xi_h$ are all roots of unity. For, then we can intersect with $x_{h+1} = \zeta_{h+1}, \ldots, x_{h+s-1} = \zeta_{h+s-1}$ for general roots of unity $\zeta_{h+1}, \ldots, \zeta_{h+s-1}$ and then with general $x_{h+s} = \xi_{h+s}$ to get rank at most $1 + (n - h - s) \leq n - r$.

Thus in the Bounded Height Conjecture it is definitely necessary to remove those special anomalous $\mathcal{Y}$ with K restricted as above. The other $\mathcal{Y}$ may not provide counterexamples. But if we remove only the special $\mathcal{Y}$, then what remains is not always Zariski-open as in Theorem 1.4. An example of this exists already for $n = 3$ given by the plane

$\mathcal{X}$ defined by $x_3 = x_1 + x_2$. It is easy to see that the anomalous curves are defined by $x_1 = \alpha_1 x_3$, $x_2 = \alpha_2 x_3$ with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ satisfying $\alpha_1 + \alpha_2 = 1$. They are special in the above sense exactly when α_1, α_2 are multiplicatively dependent; as pointed out in [4] (p.1119), this happens infinitely often but clearly only countably so. Thus what remains is not open.

Therefore if we want to claim bounded height on an open set, then probably the Bounded Height Conjecture is the most suitable candidate.

The other main result of [4] concerned finiteness of intersections. It was proved via the boundedness of the height, but this does not lead to absolutely sharp results. In fact it is not difficult to formulate a conjecture analogous to the Bounded Height Conjecture for this situation. In [6] we did this for curves $\mathcal{X}$, and Zhang had earlier also considered such things. So it seems appropriate here briefly to describe our own versions. We return now to the earlier more general context, with varieties defined over $\mathbf{C}$.

A torsion coset is by definition a coset gH with a torsion point g and an algebraic group H. We say that an irreducible subvariety $\mathcal{Y}$ of $\mathcal{X}$ is torsion-anomalous if it has positive dimension and lies in a torsion coset K of an algebraic subgroup of $\mathbf{G}_m^n$ also satisfying (1.1) or (1.2).

Now $\mathcal{X}^{ta}$ is what remains of $\mathcal{X}$ after removing all torsion-anomalous subvarieties.

We are unable to prove the analogue of Theorem 1.4 above on openness of $\mathcal{X}^{ta}$. Therefore we state:

Torsion Openness Conjecture. *Let $\mathcal{X}$ be an irreducible variety in $\mathbf{G}_m^n$ defined over $\mathbf{C}$. Then $\mathcal{X}^{ta}$ is Zariski-open in $\mathcal{X}$.*

When $\mathcal{X}$ is a curve $\mathcal{C}$ this does reduce to a triviality; but otherwise almost certainly not. When $\mathcal{X}$ is a hypersurface, then $\mathcal{X}^{ta}$ is $\mathcal{X}$ deprived of the positive-dimensional torsion cosets lying in $\mathcal{X}$. Laurent proved in Lemma 4 of [16] (p.308) that there are only finitely many maximal connected torsion cosets in $\mathcal{X}$ altogether (see also Theorems 1 and 2 of [23] (p.159), as well as Theorem 2(a) of [8] (p.336) when $\mathcal{X}$ is defined over $\overline{\mathbf{Q}}$). Therefore in this case the Torsion Openness Conjecture is true. Its non-triviality in general will be clearer after we have proceeded to the next conjecture.

Torsion Finiteness Conjecture. *Let $\mathcal{X}$ be an irreducible variety in $\mathbf{G}_m^n$ of dimension r defined over $\mathbf{C}$. Then $\mathcal{X}^{ta} \cap \mathcal{H}_{n-r-1}$ is a finite set.*

This conjecture is sharp in the sense that $\mathcal{Y} \cap \mathcal{H}_{n-r-1}$ is infinite for any torsion-anomalous subvariety $\mathcal{Y}$ of $\mathcal{X}$. Actually $\mathcal{Y} \cap \mathcal{H}_{n-r-1}$ is even Zariski-dense in $\mathcal{Y}$. As we will shortly make use of this fact, we sketch a proof as follows. We know that $\mathcal{Y}$ lies in an algebraic subgroup H of $\mathbf{G}_m^n$ of dimension $n-h<n-r+s$, where $s=\dim\mathcal{Y}\geq 1$. After an automorphism we can assume that the coordinates $x_1,\ldots,x_h$ are fixed roots of unity $\zeta_1,\ldots,\zeta_h$ on H. We can also assume that $x_{h+1},\ldots,x_{h+s}$ are algebraically independent of $\mathcal{Y}$. We can now intersect $\mathcal{Y}$ with $x_{h+1}=\zeta_{h+1},\ldots,x_{h+s}=\zeta_{h+s}$ for varying roots of unity $\zeta_{h+1},\ldots,\zeta_{h+s}$. The multiplicative rank of the coordinates is then at most $n-(h+s)<n-r$, and so we get points of $\mathcal{Y}\cap\mathcal{H}_{n-r-1}$. And as $\zeta_{h+1},\ldots,\zeta_{h+s}$ vary, these are dense as required.

When $\mathcal{X}$ is a curve $\mathcal{C}$ not lying in any torsion coset of dimension $n-1$, then $\mathcal{X}^{ta}=\mathcal{C}$. So provided $\mathcal{C}$ is defined over $\overline{\mathbf{Q}}$, the Torsion Finiteness Conjecture reduces to Conjecture A of [6] (p.2248). This Conjecture A was proved for $n=1,2,3,4,5$ as the Corollary in [6] (p.2248). When $\mathcal{C}$ is restricted further not to lie in any coset whatsoever of dimension $n-1$, it was proved for general n as Theorem 2 of [4] (p.1120). The latter result was extended to curves over $\mathbf{C}$ in the Theorem of [5] (p.451). When $\mathcal{X}$ is a hypersurface defined over $\overline{\mathbf{Q}}$, then removing all the finitely many maximal connected torsion cosets in $\mathcal{X}$ leaves us with the set $\mathcal{X}^*$ defined in [8] (p.335), which of course contains no torsion points. Hence in this case $\mathcal{X}^{ta}\cap\mathcal{H}_0$ is indeed a finite set in accordance with the Torsion Finiteness Conjecture. Nothing else has been published up to now. In a forthcoming paper [7] we shall prove the weaker assertion that $\mathcal{X}^{oa}\cap\mathcal{H}_{n-r-1}=\mathcal{X}^{oa}\cap\mathcal{H}_{n-3}$ is a finite set for any plane $\mathcal{X}$ defined over $\overline{\mathbf{Q}}$ in any $\mathbf{G}_m^n$.

Here we verify that our Theorem 1.4 implies our Theorem 1.7; that is, both the Torsion Openness Conjecture and the Torsion Finiteness Conjecture for any variety $\mathcal{X}$ of dimension $n-2$ defined over $\overline{\mathbf{Q}}$. In section 4 we checked that Theorem 1.5 implies Theorem 1.4. We leave the reader to complete the triangle with a simple proof that Theorem 1.7 implies Theorem 3.

Proof of Theorem 5.1. We first show by induction on n that there is a finite collection $\Omega=\Omega_{\mathcal{X}}$ of torsion-anomalous subvarieties $\mathcal{Y}$ of $\mathcal{X}$ such that the intersection of $\mathcal{H}_1$ with $\mathcal{X}$ deprived of the members of Ω is finite. This is trivial for $n=2$.

Thus suppose $n\geq 3$. We note in Theorem 1.4 that

$$\dim T\ \leq\ 1+\dim(\mathcal{X}\cap T)\ \leq\ 1+\dim\mathcal{X}\ =\ n-1$$

for each T in Ψ. Thus by enlarging T if necessary we can assume that it has dimension $n-1$ and that $\mathcal{X}\cap\mathcal{H}_1$ is contained in the union of the $(\mathcal{X}\cap T)\cap\mathcal{H}_1$.

Next, we claim that we can also assume that every component of each $\mathcal{X}\cap T$ has dimension $n-3$. For, after an automorphism we can suppose that T is defined by $x_n=\zeta_n$ for a root of unity ζ_n. If the projection π of $\mathcal{X}$ to the last coordinate is not dominant, then

x_n would be constant on $\mathcal{X}$. But then $x_n = \zeta_n$ on $\mathcal{X}$. This would mean that $\mathcal{X}$ itself is torsion-anomalous, and in this case the induction statement above is trivially true with a single $\mathcal{Y} = \mathcal{X}$. So we can suppose that π is dominant. Now FDT(a) shows that every component of $\mathcal{X} \cap T$ has dimension at least $(n-2)-1 = n-3$. However, if some component had dimension $n-2$, then $\mathcal{X}$ would be contained in T and so again $\mathcal{X}$ would be torsion-anomalous. This proves the claim above.

Now the projection $\mathcal{X}'$ of a component of $\mathcal{X} \cap T$ to $\mathbf{G}_m^{n-1}$ has dimension $n-3 = (n-1)-2$, and so the induction hypothesis gives a finite collection Ω' of torsion-anomalous subvarieties $\mathcal{Y}'$ of $\mathcal{X}'$ such that the intersection of $\mathcal{H}_1$ with $\mathcal{X}'$ deprived of the members of Ω' is finite. It is not difficult to see that each $\mathcal{Y}' \times \{\zeta_n\}$ here is also torsion-anomalous in $\mathcal{X}$ in $\mathbf{G}_m^n$. For $\dim \mathcal{Y}' \geq 1$ and $\mathcal{Y}'$ lies in an algebraic subgroup H of $\mathbf{G}_m^{n-1}$ satisfying

$$\dim \mathcal{Y}' \;\geq\; 1 + \dim \mathcal{X}' + \dim H - (n-1) \;=\; \dim H - 1.$$

This is

$$\dim(\mathcal{Y}' \times \{\zeta_n\}) \;\geq\; \dim H - 1 \;=\; 1 + \dim \mathcal{X} + \dim(H \times \{\zeta_n\}) - n.$$

Thus, indeed, $\mathcal{Y}' \times \{\zeta_n\}$ is torsion-anomalous in $\mathcal{X}$ in $\mathbf{G}_m^n$. And so to get $\Omega_{\mathcal{X}}$ above, it suffices to take the union of these. This establishes the induction statement above.

To finish off, we say that a torsion-anomalous subvariety of $\mathcal{X}$ is maximal if it is not contained in a strictly larger torsion-anomalous subvariety of $\mathcal{X}$. Let $\mathcal{Y}_0$ be any torsion-anomalous subvariety of $\mathcal{X}$. We observed above that $\mathcal{Y}_0 \cap \mathcal{H}_1$ is dense in $\mathcal{Y}_0$. It follows that $\mathcal{Y}_0$ lies in the union of the $\mathcal{Y}$ in Ω. Thus if $\mathcal{Y}_0$ was maximal torsion-anomalous, then it must be one of these $\mathcal{Y}$. This shows that there are only finitely many maximal torsion-anomalous subvarieties of $\mathcal{X}$; and now the proof of Theorem 1.7 is easily completed. ■

The probable difficulty of the Torsion Openness Conjecture in general is due to the fact that it implies the Torsion Finiteness Conjecture. To be more precise, the Torsion Openness Conjecture for an arbitrary variety $\mathcal{X}$ and also for $\mathcal{X} \times \mathbf{G}_m$ implies the Torsion Finiteness Conjecture for $\mathcal{X}$. To see this, we first show that

$$(\mathcal{X} \times \mathbf{G}_m)^{ta} \;=\; U \times \mathbf{G}_m, \tag{5.1}$$

where U is what remains of $\mathcal{X}^{ta}$ after removing $\mathcal{X}^{ta} \cap \mathcal{H}_{n-r-1}$.

Namely, let $\hat{\mathcal{Y}}$ in $\mathbf{G}_m^{n+1}$ be something removed from the left-hand side of (5.1). Thus $\hat{\mathcal{Y}}$ is a torsion-anomalous subvariety of dimension say $s \geq 1$ in $\mathcal{X} \times \mathbf{G}_m$. Then $\hat{\mathcal{Y}}$ lies in a torsion coset $\hat{K}$ in $\mathbf{G}_m^{n+1}$ satisfying

$$\dim \hat{K} \leq (n+1)-(r+1)+s-1 = n-r+s-1.$$

Projecting $\hat{\mathcal{Y}}$ down to $\mathbf{G}_m^n$ gives a subvariety $\mathcal{Y}$ of $\mathcal{X}$ lying in a torsion coset $K = \pi(\hat{K})$ of $\mathbf{G}_m^n$. If $\dim \mathcal{Y} = s$ then

$$\dim K \leq \dim \hat{K} \leq n-r+s-1$$

so that $\mathcal{Y}$ is torsion-anomalous in $\mathcal{X}$. Thus $\mathcal{Y} \times \mathbf{G}_m$ is removed from the right-hand side of (5.1).

If $1 \leq \dim \mathcal{Y} < s$, then $\dim \mathcal{Y} = s-1 \geq 1$ and $\hat{\mathcal{Y}} = \mathcal{Y} \times \mathbf{G}_m$; this forces $\hat{K} = K \times \mathbf{G}_m$ so

$$\dim K = \dim \hat{K} - 1 \leq n-r+s-2 = n-r+(s-1)-1$$

and we reach the same conclusions.

Finally, if $\dim \mathcal{Y} = 0$, then $\hat{\mathcal{Y}} = \{x\} \times \mathbf{G}_m$ for a point x, so $s = 1$ and again $\hat{K} = K \times \mathbf{G}_m$; but now x must lie in K of dimension

$$\dim K \leq n-r+s-2 = n-r-1$$

So x lies in $\mathcal{H}_{n-r-1}$, and here too $\hat{\mathcal{Y}} = \{x\} \times \mathbf{G}_m$ is removed from the right-hand side of (5.1).

All this proves that the left-hand side of (5.1) contains the right-hand side. To see the opposite inclusion, note that what is removed from the right-hand side has the form $\hat{\mathcal{Y}} = \mathcal{Y} \times \mathbf{G}_m$, again of dimension say $s \geq 1$, either for $\mathcal{Y}$ in a torsion coset K satisfying

$$\dim K \leq n-r+(s-1)-1 = n-r+s-2,$$

or for $\mathcal{Y} = \{x\}$ with x in $\mathcal{X}^{ta} \cap \mathcal{H}_{n-r-1}$. In the first case $\hat{\mathcal{Y}}$ lies in the torsion coset $\hat{K} = K \times \mathbf{G}_m$ satisfying

$$\dim \hat{K} \leq (n-r+s-2)+1 = (n+1)-(r+1)+s-1.$$

So $\hat{\mathcal{Y}}$ is torsion-anomalous in $\mathcal{X} \times \mathbf{G}_m$, and is therefore removed from the left-hand side of (5.1) too. In the second case $\hat{\mathcal{Y}}$ lies in a torsion coset of dimension at most $n-r = (n+1)-(r+1)+s-1$, and we reach the same conclusions. Thus indeed (5.1) holds.

Consequently the Torsion Openness Conjecture for $\mathcal{X} \times \mathbf{G}_m$ implies that U is open in $\mathcal{X}$; here $U = \mathcal{X}^{ta} \setminus X$ for $X = \mathcal{X}^{ta} \cap \mathcal{H}_{n-r-1}$. However, $\mathcal{X}^{ta}$ is also open. And $\mathcal{X}^{ta} \cap \mathcal{H}_{n-r}$ is at most countable (any positive-dimensional component would be torsion-anomalous), and so certainly X is at most countable. It follows that X must be finite, in accordance with the Torsion Finiteness Conjecture for $\mathcal{X}$.

We now show that these two conjectures together (which in some sense are equivalent to just the Torsion Openness Conjecture) imply some conjectures of Boris Zilber for $\mathbf{G}_m^n$ (he also treats semiabelian varieties, on which we shall shortly comment). His Conjecture 1 of [25] (p.29) is stated just for varieties $\mathcal{X}$ defined over the rationals $\mathbf{Q}$ (but, as remarked there, immediately implies the version over $\overline{\mathbf{Q}}$). There is also a version (Conjecture 1 with parameters) that could be interpreted to involve varieties over $\mathbf{C}$; this fits better with our present viewpoint. Let us therefore make a statement in the following form, where for consistency we continue to insist that tori are connected algebraic subgroups (unlike [25], where they are simply general cosets).

Conjecture 5.2. (Zilber) Let $\mathcal{X}$ be an irreducible variety in $\mathbf{G}_m^n$ defined over $\mathbf{C}$. Then there is a finite collection $\Omega = \Omega_{\mathcal{X}}$ of translates T of tori of dimension at most $n-1$ by torsion points such that for every torsion coset K and every component $\mathcal{Y}$ of $\mathcal{X} \cap K$ satisfying

$$\dim \mathcal{Y} \; > \; \dim \mathcal{X} + \dim K - n$$

one has $\mathcal{Y} \subseteq T$ for some T in Ω. □

In fact our Torsion Openness Conjecture for $\mathcal{X}$ and our Torsion Finiteness Conjecture for $\mathcal{X}$ together imply Zilber's Conjecture for $\mathcal{X}$ of dimension r. Clearly $\mathcal{X}^{ta}$ is obtained from $\mathcal{X}$ by removing all maximal torsion-anomalous subvarieties. Because $\mathbf{G}_m^n$ has only countably many algebraic subgroups, there are at most countably many such maximal torsion-anomalous subvarieties. Thus the openness of $\mathcal{X}^{ta}$ implies that there are in fact at most finitely many maximal torsion-anomalous subvarieties. Each of these is contained in a translate T of a torus by a torsion point satisfying

$$\dim T \; < \; n - \dim \mathcal{X} + \dim \mathcal{Y} \; \leq \; n. \tag{5.2}$$

Now take a $\mathcal{Y}$ as in Zilber's Conjecture. If $\dim \mathcal{Y} > 0$, then $\mathcal{Y}$ is torsion-anomalous. So for this case it suffices to put the T in (5.2) into Ω. If $\dim \mathcal{Y} = 0$, then $\mathcal{Y}$ is a point lying in an algebraic subgroup of dimension at most $n-r-1$, so in $\mathcal{X} \cap \mathcal{H}_{n-r-1}$. If $\mathcal{Y}$ is not in $\mathcal{X}^{ta}$, then it lies in a maximal torsion-anomalous subvariety as before and we are done.

Otherwise $\mathcal{Y}$ lies in the finite set $\mathcal{X}^{ta} \cap \mathcal{H}_{n-r-1}$. Each of these points certainly lies in a T as above with dimension at most $n-1$, and so it suffices to enlarge Ω accordingly.

As suggested by Richard Pink, one can also check that Zilber's Conjecture implies our Torsion Openness Conjecture, provided one deals with all varieties at once. We omit the details here, because in a note presently being prepared, we will establish among other things a series of such implications and equivalences in a slightly stronger form.

We next show that our two conjectures together imply some conjectures of Pink for $\mathbf{G}_m^n$ (he too treats semiabelian varieties and even mixed Shimura varieties, on which we shall also shortly comment). His Conjecture 5.1 of [18] (p.6), restricted to the multiplicative case, says in our language the following.

Conjecture 5.3. (Pink) Let $\mathcal{X}$ be an irreducible variety in $\mathbf{G}_m^n$ of dimension r defined over $\mathbf{C}$, which is not contained in an algebraic subgroup of dimension at most $n-1$. Then $\mathcal{X} \cap \mathcal{H}_{n-r-1}$ is not Zariski-dense in $\mathcal{X}$. □

In fact our Torsion Openness Conjecture for $\mathcal{X}$ and our Torsion Finiteness Conjecture for $\mathcal{X}$ together imply the above Conjecture for $\mathcal{X}$. For, certainly $\mathcal{X}^{ta} \cap \mathcal{H}_{n-r-1}$ is a finite set, so it suffices to deal with the finitely many maximal torsion-anomalous subvarieties $\mathcal{Y}$ referred to above. For each $\mathcal{Y} \neq \mathcal{X}$ there is no problem; and if $\mathcal{Y} = \mathcal{X}$, then the torsion analogue of (1.1) shows that $\mathcal{X}$ is contained in an algebraic subgroup of dimension at most $n-1$.

6 Generalizations

We append here some comments about contexts more general than the multiplicative group variety $\mathbf{G}_m^n$. Of course this is a very special case of a semiabelian variety S (and these include abelian varieties). If $\mathcal{X}$ is a closed subvariety of S, then the definition of $\mathcal{X}^{oa}$ is just as in (1.1) or (1.2) with translates K of algebraic subgroups of S and $n = \dim S$. The natural analogue of our Theorem 1.4 can then be proved following the lines of this paper with relatively routine modifications. We can also formulate a Bounded Height Conjecture. However, the analogues of our Theorems 1.4 and 1.5 cannot be proved without some extra complex multiplication hypotheses, owing to the use of the height lower bounds (4.2).

Likewise, there are natural analogues of our Torsion Openness Conjecture and our Torsion Finiteness Conjecture. In fact for abelian varieties Rémond has independently posed a more general problem as the Question in [19] (p.526); the choice $1 + \dim \mathcal{X}$ for his parameter r (not the same as ours) yields the Torsion Finiteness Conjecture in this case. And already Zilber had generalized a version of his own conjecture to the semi-abelian case as Conjecture 2 of [25] (p.31).

For the case of mixed Shimura varieties as proposed by Pink [18], relatively little is known. We confine ourselves here to pointing out that the analogue of our Bounded Height Conjecture fails for subvarieties of Shimura varieties, even for $\mathcal{X} = \mathbf{P}_1$. Here roots of unity ζ are replaced by values $\sigma = j(\tau)$ of the elliptic modular function j at complex quadratic τ. It is known that their absolute heights $h(\sigma)$ are unbounded; see for example Théorème 1 (p.360) of the article [10] by Colmez. In fact this is true even on a fixed Hecke orbit, and even for specific examples like $\tau = p\sqrt{-1}$ for prime p. To see this, note that σ then satisfies the equation $F_p(\sigma, 1728) = 0$, where $F_p(X, Y)$ in $\mathbf{Z}[X, Y]$ is the modular transformation polynomial of order p. Now $F = F_p(X, 1728)$ has degree $p + 1$, and the work of Paula Cohen implies that this monic polynomial has a coefficient bigger than p^{5p} in absolute value for large p; see the Proposition of [9] (p.390). Further, it is known that σ has degree exactly $d = \frac{1}{2}(p + 1)$ when $p \equiv 3 \pmod 4$; see for example Theorem 7 of [15] (p.95) and Theorem 5 of [15] (p.133). As $\sigma = j(p\sqrt{-1}) = j(\frac{\sqrt{-1}}{p})$ occurs twice in the factorization of F, we deduce in this case that $F = G^2$ for irreducible G in $\mathbf{Z}[X]$. The required assertion now follows from classical comparisons between the Mahler measure M and the height; for example, using the right-hand inequality in equation (38) of Corollary 11 of [22] (p.248) we see that the relative height

$$dh(\sigma) = \log M(G) = \frac{1}{2}\log M(F) \geq \frac{1}{2}\log(p^{5p}/B),$$

where B is the binomial coefficient $\binom{p+1}{d} \leq 2^{p+1}$. So $h(j(p\sqrt{-1})) \geq 4\log p$ for all large $p \equiv 3 \pmod 4$.

Corrigendum by Umberto Zannier, to *Appendix by Umberto Zannier* in [24].

I avail myself of this opportunity to point out an inaccuracy in the exposition [24] that I wrote for the book [22], of the proof (obtained jointly with Bombieri) of Theorem 2 therein; I also indicate how this gap may be immediately repaired. (Naturally, the present results in any case would provide another proof of the relevant theorems.)

The gap occurred just at the final lines of the proof: at line 4 of p. 538 of [22] the inequalities "$g \leq \deg(X)\deg(G(v_1)) \leq c_7 l$" appear. The first one is generally incorrect (if $\dim X > 2$). Inspection of the few preceding lines (the last three lines of p. 537 suffice) immediately shows that a correct inequality is "$g \leq \deg(X)\deg(G(v_1, \dots, v_m, w_1, \dots, w_r))$" where the vectors v_i, w_j are defined previously therein. In turn, the right side may be estimated as $c_7 l^e$ where e is a suitable number depending only on X, which can be extracted from the previous steps of that proof.

The remaining part of the proof (actually just a few lines) may be left unchanged; one has only to choose the number c_{14} sufficiently large also in terms of e (inspection shows that there is no constraint against this to be done).

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